Lower Bounds for Line Searching Robots – Some Faulty

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[Richard E. Bellman'63] Suppose that we know that **a particle is located** in the interval $(x, x+dx)$, **somewhere along the real line** $-\infty < x < \infty$ **with a probability density function** $g(x)$. We start at some initial point x_0 and can move in either direction. What policy minimizes the expected time required to find the particle,

[Anatole Beck'64]: **A man in an automobile searches for another man** who is located at some point of a certain road. He starts at a given point and **knows in advance the probability** that the second man is at any given point of the road. Since the man being sought might be in either direction . . .

Anatole Beck wrote quite a few papers on the topic:

On the linear search problem (1964)

More on the linear search problem (1965)

Yet more on the linear search problem (1970)

The return of the linear search problem (1973)

Son of the linear search problem (1984)

The linear search problem rides again (1986)

The revenge of the linear search problem (1992)

Computer Science rediscovers the problem

Cow Path Problem – Cow at a Fence Problem

[Baeza-Yates,Culberson,Rawlins'88]: **A cow comes to an infinitely long straight fence.** The cow knows that there is a gate in the fence, and she wants to get to the other side. Unfortunately, she doesn't know where the gate is located ...

We start at 0 (origin), move at constant speed 1, and want to find the target at x, $|x| \geq 1$, in time at most $\lambda|x|$, λ as small as possible. (λ-competitive)

[Beck,Newman'70] $\lambda = 9$ is tight.

rand.4.591–comp.[Kao,Reif,Tate'94]

Note: Without "$|x| \geq 1$" no competitive ratio is possible.

If we move ε in one direction, the adversary places the gate at $\frac{\varepsilon}{1'000'000}$ in the other direction.

A Natural Problem Appearing in Numerous Scenarios

A robot trying to get around an obstacle.

k-server problem. [Fiat,Rabani,Ravid‘91]

Different algorithms available to solve an instance, which one to choose. (Hybrid algorithms [Kao,Ma,Sipser,Yin‘01])

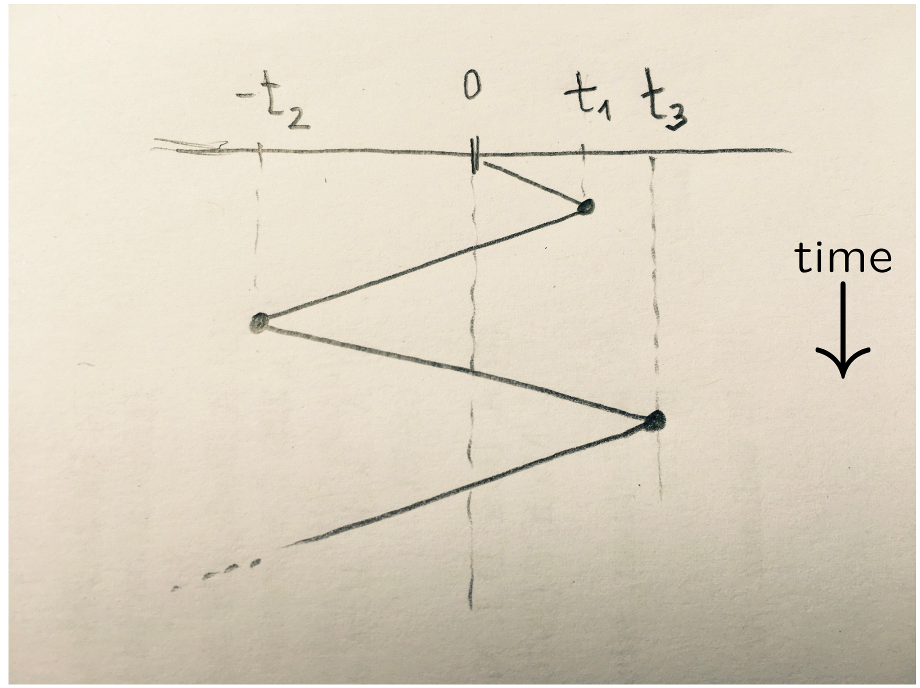
We can restrict ourselves to **strategies** which

move to t_1, turn and

move to $-t_2$, turn and

move to t_3, turn and

...



for $T = (t_1, t_2, t_3, \ldots) \in (\mathbb{R}_+)^{\mathbb{N}}$.

$(1, 2, 4, \ldots, 2^i, \ldots)$ gives 9-competitive strategy.

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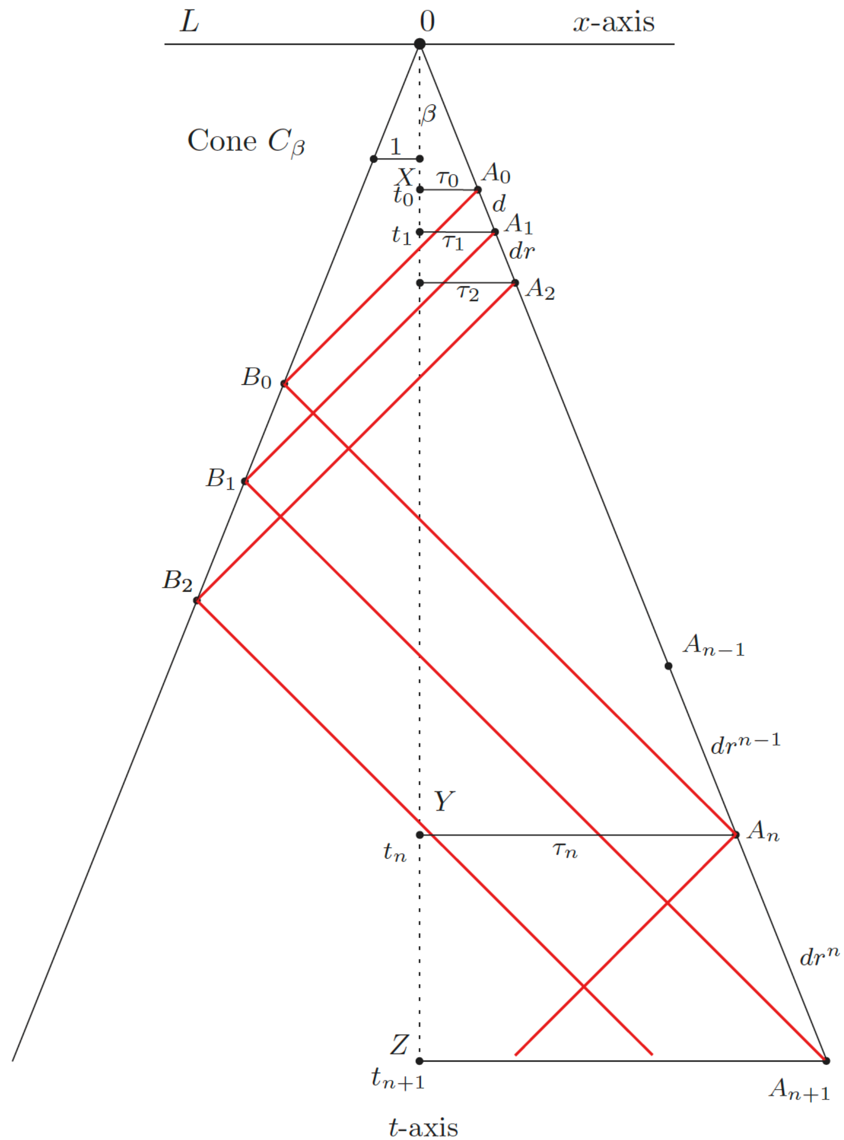
[Czyzowitz,Kranakis,Krizanc,Narayanan,Opatrny PODC'16] some of the robots are faulty (i.e. fail to report the target despite of hitting it).

They suggest a strategy that, e.g. given $k = 3$ robots, $f = 1$ faulty, finds the target at x in time $\lambda|x|$, $\lambda \approx 5.24$.

Is this tight? (They show a lower bound of $\lambda > 3.76$.)

Strategy from [CKKNO‘16]:



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Given k robots, f faulty, $f < k < 2(f+1)$ we provide lower bounds matching the upper bounds of [CKKNO'16].

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In order to be λ-competitive, we have to make sure that every x, $|x| \in \mathbb{R}_{\geq 1}$, is visited by at $\geq f+1$ robots in time $\leq \lambda|x|$; otherwise the adversary places the target there and chooses the first f robots arriving to be faulty.

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That means that for all $x \in \mathbb{R}_{\geq 1}$, at least

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Lemma: A strategy for k robots, f faulty, is λ-competitive $\Rightarrow$ the strategy (λ, s)-covers $\mathbb{R}_{\geq 1}$.

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New Goal: Given k and s, provide a lower bound on λ, for (λ, s)-covering of $\mathbb{R}_{\geq 1}$ with k robots.

Result

Theorem: (λ, s)-covering with k-robots is impossible if

$$\lambda < 2\sqrt[k]{\frac{(k+s)^{k+s}}{s^s k^k}} + 1 \ .$$

Without proof (but easy to show): If (λ, s)-covering is possible, then with strategies

$$T^{(r)} = (t_1^{(r)}, t_2^{(r)}, t_3^{(r)}, \ldots), \quad r = 1, 2, 3, \ldots, k$$

where $1 \leq t_1^{(r)} \leq t_2^{(r)} \leq t_3^{(r)} \leq \cdots$.

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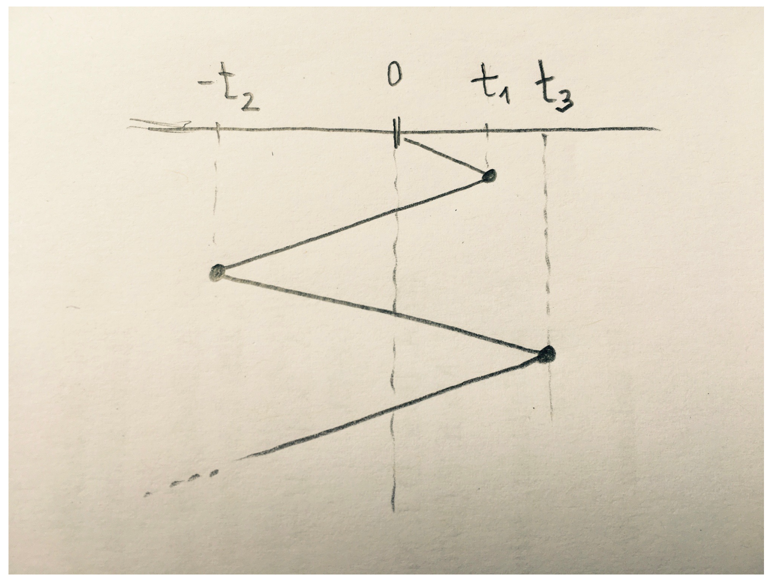
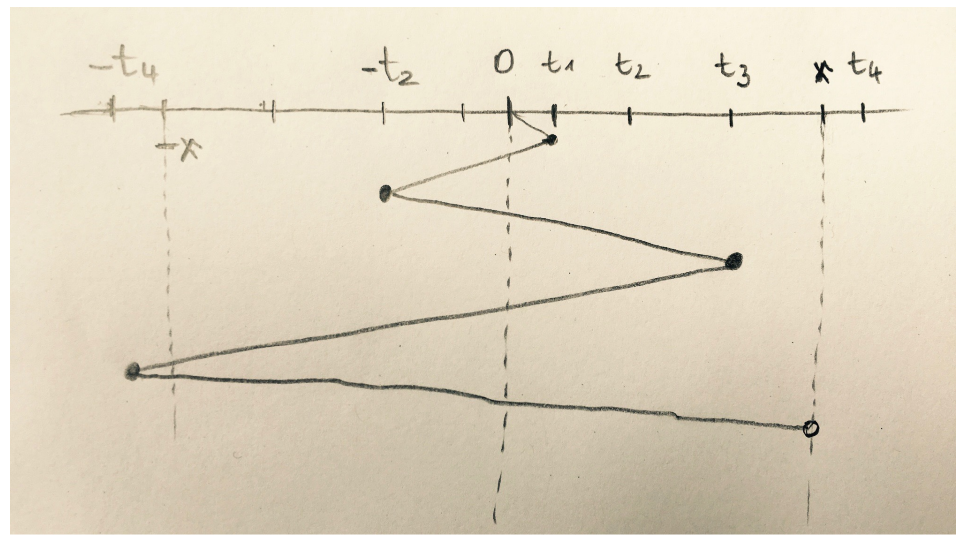


Given a strategy $T = (t_1, t_2, t_3, \ldots)$ and x with $t_{i-1} < x < t_i$, then $\{-x, x\}$ is visited in time

$$2(t_1 + t_2 + \cdots + t_i) + x$$



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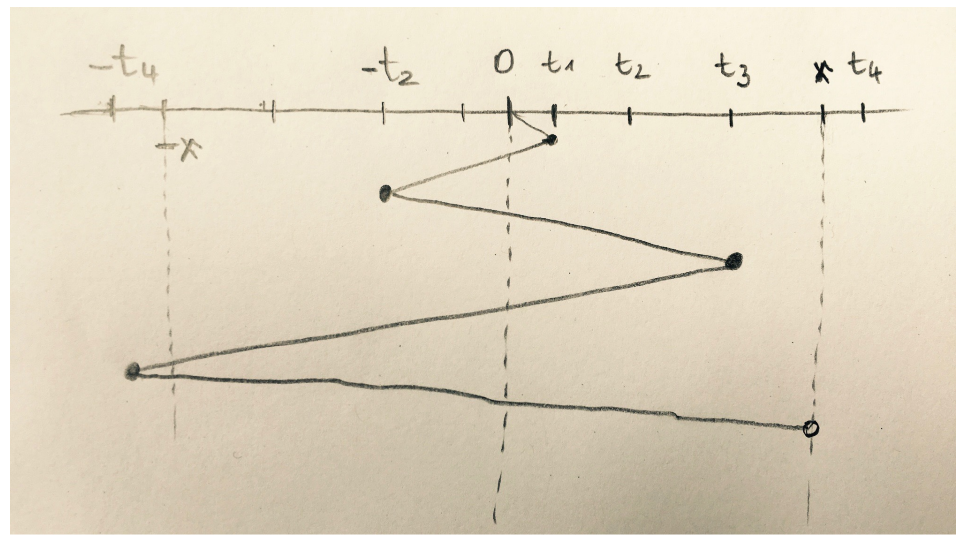

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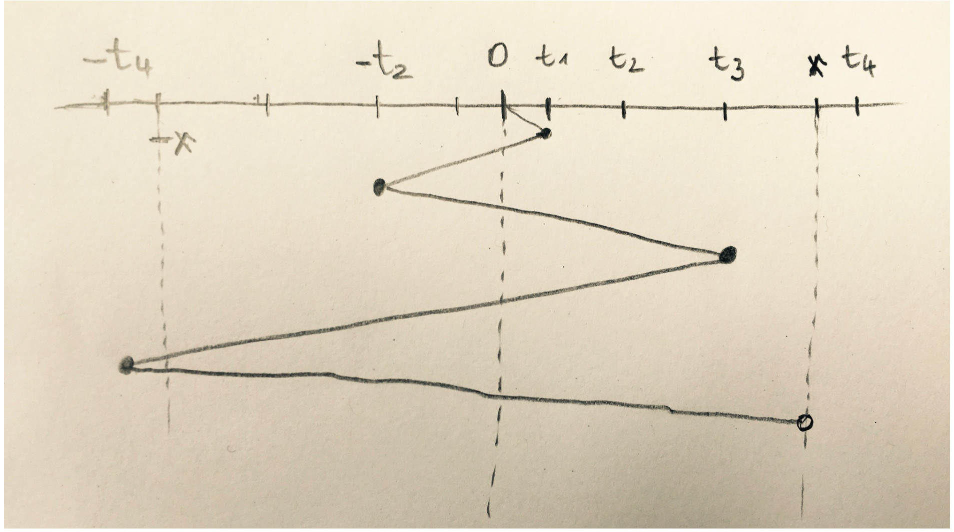
That is, x is λ-covered iff

$$2(t_1 + \cdots + t_i) + x \leq \lambda x$$

$$\Leftrightarrow \quad x \geq \frac{1}{\mu}(t_1 + \cdots + t_i),$$

$$\mu := \tfrac{\lambda - 1}{2}$$



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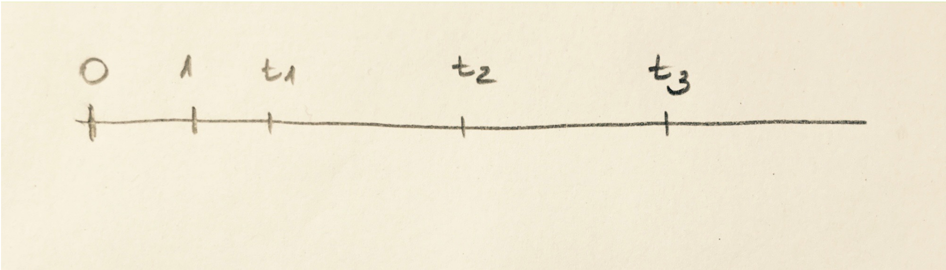

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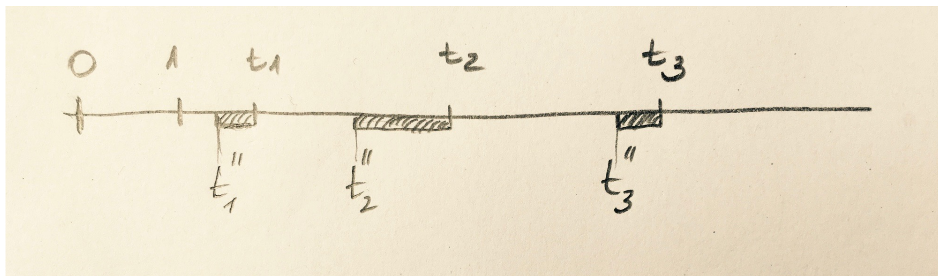
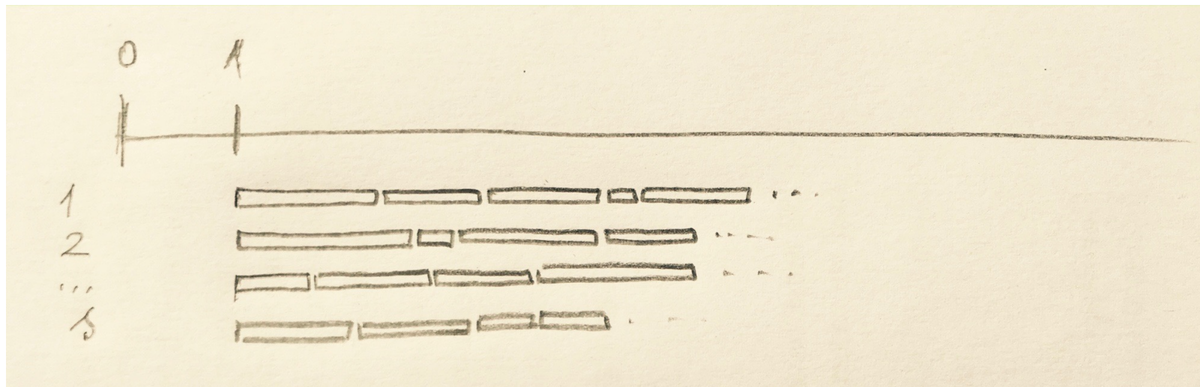


Robot r with strategy $T^{(r)} = T = (t_1, t_2, t_3, \ldots)$ λ-covers exactly

$$\bigcup_i [t_i'', t_i]$$

We choose values t_i', $t_i'' \leq t_i' \leq t_i$, such that the intervals $(t_i', t_i]$ (of all robots) cover every $x \in \mathbb{R}_{>1}$ exactly s times.

$\left(t_i^{(r)'}, t_i^{(r)}\right]$, $i \in \mathbb{N}$, $r = 1, 2 \ldots, k$, cover $\mathbb{R}_{>1}$ exactly s times.



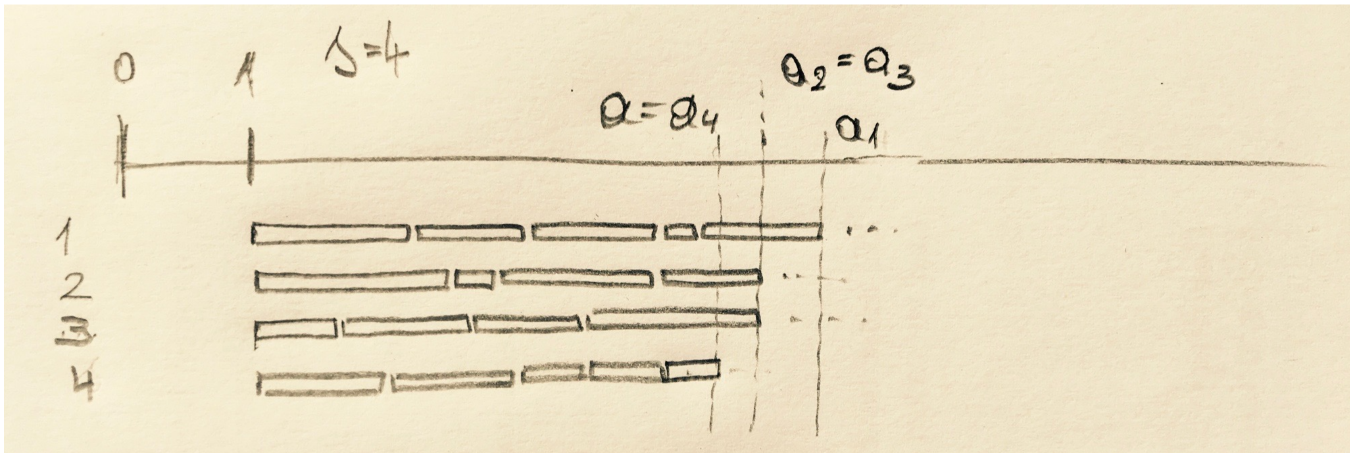
$$t_i' \geq \frac{1}{\mu}(t_1 + \cdots + t_i)$$

$$\Leftrightarrow \quad t_1 + \cdots + t_i \leq \mu t_i'$$

$$\Leftrightarrow \quad t_i \leq \mu t_i' - (t_1 + \cdots + t_{i-1})$$

Collect these intervals in a common sequence, sorted by left endpoints (ties broken arbitrarily).

Consider a prefix $\mathcal{P}$ of this sequence of intervals



$\mathcal{P}$ covers s times up to a point $a = a(\mathcal{P})$ and there are numbers

$$a = a_s \leq a_{s-1} \leq \cdots \leq a_1$$

such that $\mathcal{P}$ covers j times for $(a_{j+1}, a_j]$ and not at all for (a_1, ∞).

$$A(\mathcal{P}) := \{a_s, a_{s-1}, \ldots, a_1\}$$

a multiset describing the “current covering situation”.

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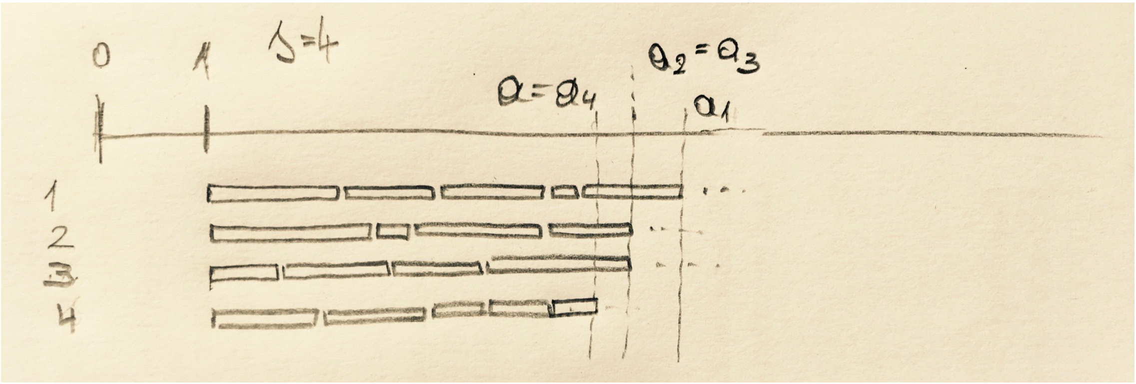

$$
\begin{aligned}
& t'_i \geq \frac{1}{\mu}(t_1 + \cdots + t_i) \\
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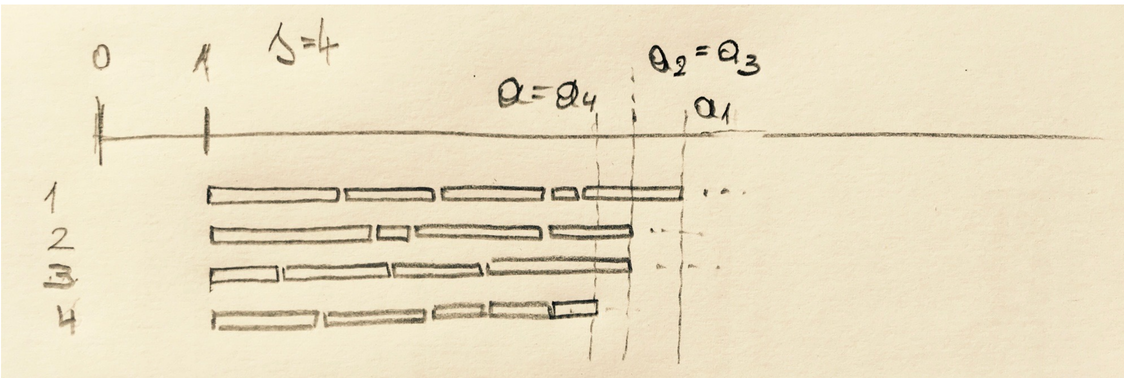

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The load of robot r in $\mathcal{P}$:

$L^{(r)}(\mathcal{P}) := t_1 + \ldots + t_{i_r}$, where $(t'_{i_r}, t'_{i_r}]$ is r's last interval in $\mathcal{P}$

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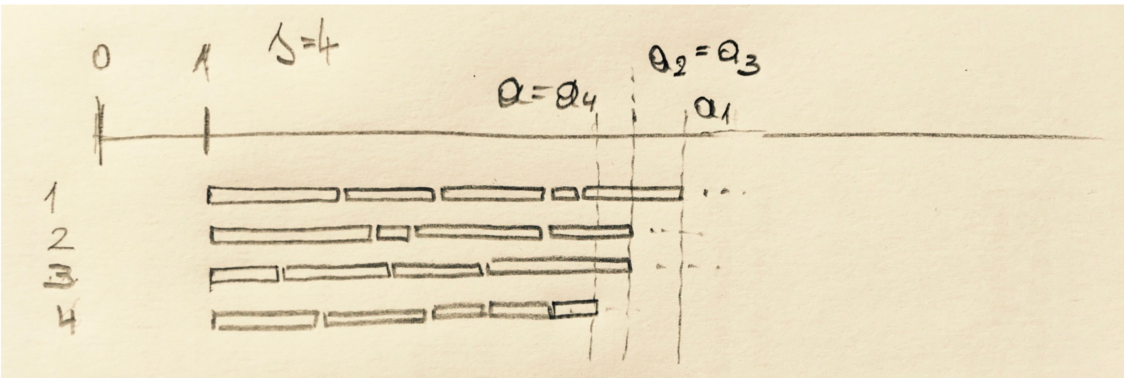

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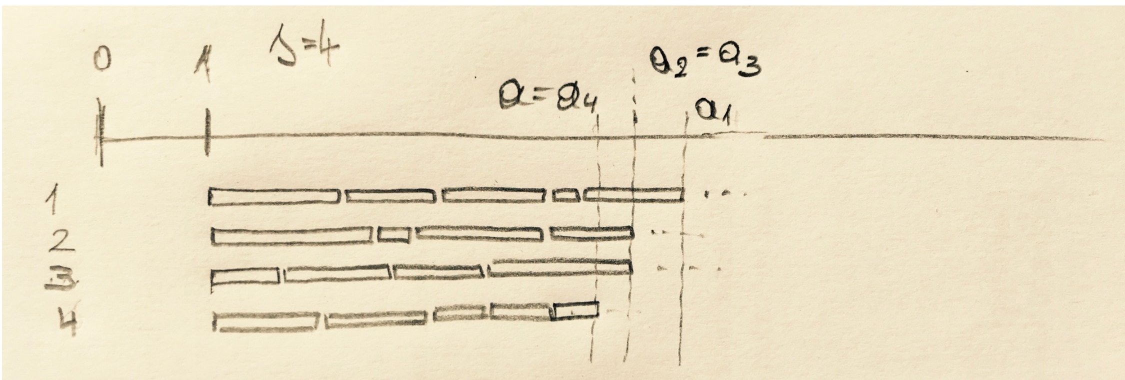

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Intuition: Large a_i's and small $L^{(r)}(\mathcal{P})$'s are good for progress!

Compress status quo in a function

$$f(\mathcal{P}) := \prod_{r=1}^{k} \frac{\left(L^{(r)}(\mathcal{P})\right)^s}{\prod_{y \in A(\mathcal{P})} y}$$

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$$a \longrightarrow \quad \left(t_{i_{r^*+1}}^{(r^*)\ \prime}, t_{i_{r^*+1}}\right] \quad \longleftarrow \mu^* a - L^{(r^*)}(\mathcal{P}), \quad 0 < \mu^* \le \mu$$

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$$\frac{f(\mathcal{P}^+)}{f(\mathcal{P})} = \frac{a^k}{\left(L^{(r^*)}(\mathcal{P})\right)^s} \cdot \frac{(\mu^* a)^s}{\left(\mu^* a - L^{(r^*)}(\mathcal{P})\right)^k} = \frac{\mu^{*s}}{x^s (\mu^* - x)^k}$$

$$x := \frac{L^{(r^*)}(\mathcal{P})}{a},\ 0 < \mu^* \leq \mu$$

By simple high school calculus . . .

For $0 < x < \mu^* \leq \mu$,

$$\frac{\mu^{*s}}{x^s(\mu^* - x)^k} \geq \frac{(k+s)^{k+s}}{s^s k^k \mu^{*k}}$$

and thus

$$\frac{\mu^{*s}}{x^s(\mu^* - x)^k} \geq \delta$$

for $\delta := \frac{(k+s)^{k+s}}{s^s k^k \mu^k} > 1$, provided $\mu < \sqrt[k]{\frac{(k+s)^{k+s}}{s^s k^k}}$.

If $\frac{\mu^{*s}}{x^s(\mu^* - x)^k} \geq \delta > 1$, then $f(\mathcal{P})$ is unbounded — contradiction.

Hence, $\mu < \sqrt[k]{\frac{(k+s)^{k+s}}{s^s k^k}}$ is impossible. Recall $\lambda = 2\mu + 1$.

Theorem: (λ, s)-covering with k-robots is impossible if

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and thus λ-competitive searching with k robots, f faulty, if $s = 2(f+1) - k$.

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Generalization: We can solve the m-ray case. Some particular cases were asked by several groups of researchers.