Lower Bounds for
Line Searching Robots – Some Faulty

Andrey Kupavskii
University of Birmingham, Birmingham

with **Emo Welzl**, ETH Zürich

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[Richard E. Bellman‘63] Suppose that we know that **a particle is located** in the interval $(x, x+dx)$, **somewhere along the real line** $-\infty < x < \infty$ **with a probability density function** $g(x)$. We start at some initial point x_0 and can move in either direction. What policy minimizes the expected time required to find the particle,

[Anatole Beck‘64]: **A man in an automobile searches for another man** who is located at some point of a certain road. He starts at a given point and **knows in advance the probability** that the second man is at any given point of the road. Since the man being sought might be in either direction . . .

Anatole Beck wrote quite a few papers on the topic:

On the linear search problem (1964)

More on the linear search problem (1965)

Yet more on the linear search problem (1970)

The return of the linear search problem (1973)

Son of the linear search problem (1984)

The linear search problem rides again (1986)

The revenge of the linear search problem (1992)

Computer Science rediscovers the problem

Cow Path Problem – Cow at a Fence Problem

[Baeza-Yates,Culberson,Rawlins'88]: **A cow comes to an infinitely long straight fence.** The cow knows that there is a gate in the fence, and she wants to get to the other side. Unfortunately, she doesn't know where the gate is located . . .

We start at 0 (origin), move at constant speed 1, and want to find the target at x, $|x| \geq 1$, in time at most $\lambda|x|$, λ as small as possible. (λ-competitive)

[Beck,Newman'70] $\lambda = 9$ is tight.

rand.4.591–comp.[Kao,Reif,Tate'94]

Note: Without "$|x| \geq 1$" no competitive ratio is possible.

If we move ε in one direction, the adversary places the gate at $\frac{\varepsilon}{1'000'000}$ in the other direction.

A Natural Problem Appearing in Numerous Scenarios

A robot trying to get around an obstacle.

k-server problem. [Fiat,Rabani,Ravid‘91]

Different algorithms available to solve an instance, which one to choose. (Hybrid algorithms [Kao,Ma,Sipser,Yin‘01])

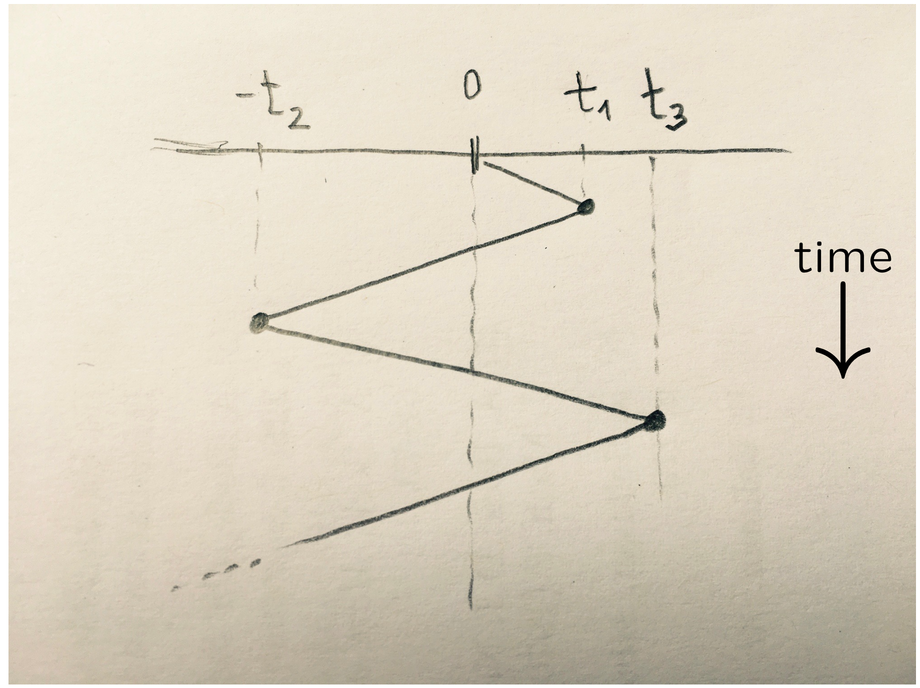
We can restrict ourselves to **strategies** which

move to t_1, turn and

move to $-t_2$, turn and

move to t_3, turn and

...



for $T = (t_1, t_2, t_3, \ldots) \in (\mathbb{R}_+)^{\mathbb{N}}$.

$(1, 2, 4, \ldots, 2^i, \ldots)$ gives 9-competitive strategy.

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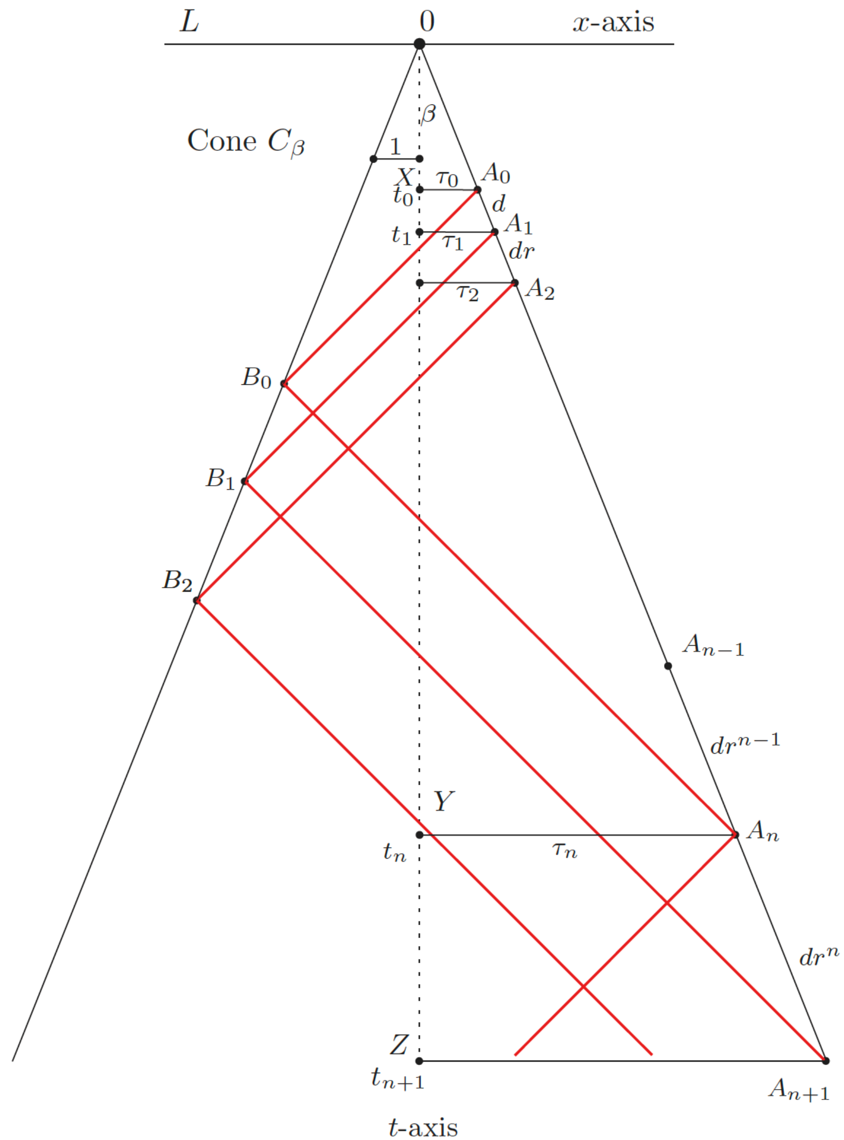
[Czyzowitz,Kranakis,Krizanc,Narayanan,Opatrny PODC'16] some of the robots are faulty (i.e. fail to report the target despite of hitting it).

They suggest a strategy that, e.g. given $k = 3$ robots, $f = 1$ faulty, finds the target at x in time $\lambda|x|$, $\lambda \approx 5.24$.

Is this tight? (They show a lower bound of $\lambda > 3.76$.)

Strategy from [CKKNO‘16]:



Related, older problem

Given k robots and m rays emanating from 0, $k < m$, find an unknown target hidden on one of the rays with best competitive ratio.

Three groups of researchers asked for the best competitive ratio: Baeza-Yates, Culberson, and Rawlins; Kao, Ma, Sipser, and Yin; Bernstein, Finkelstein, and Zilberstein.

The latter resolved the problem for a nice class of “cyclic” strategies.

Can also consider the faulty version.

Searching $\rightarrow$ Covering

Assume we have m rays and f faulty robots. Then each point of each ray must be covered $f+1$ times before we are sure that the target is not there.

Our Contribution

These problems can be relaxed to the following setting:

One-ray cover with returns (ORC): The goal is to cover $\mathbb{R}_{\geq 1}$. The robot (robots) starts at 0 and moves with unit speed along the ray $\mathbb{R}_{\geq 0}$. One robot may cover a point multiple times, but different coverings are only counted if the robot visited 0 in between.

Instead of covering m rays $f+1$ times, we cover 1 ray $q := m(f+1)$ times in the ORC setting. (The number of robots used stays the same.)

Our Contribution

For k robots and any q we provide matching lower bounds for the ORC problem. The resulting lower bounds are also matching lower bounds for the m-ray f-faulty k-robot search.

Thus, we resolve the problem from [CKKNO‘16], as well as the no-faulty m-ray cover problem by the three groups of authors.

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Theorem (AK, Welzl) Fix $q > k$ and put $\rho := \frac{q}{k}$. Then the best possible competitive problem for the q-fold ORC cover with k robots is

$$2\frac{\rho^\rho}{(\rho-1)^{\rho-1}} + 1. \qquad (1)$$

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In order to be λ-competitive, we have to make sure that every x, $|x| \in \mathbb{R}_{\geq 1}$ is covered q times in time $\leq \lambda|x|$. A robot *λ-covers* a point x, if it visits x within time λx.

Round: the time a robot spends between two consecutive visits of 0.

Can assume: In each round each robot turns once at a point t and λ-covers some non-empty interval $[x, t]$.

If the turning point of a robot in round i is t_i, then in the $i+1$-st round the point x is λ-covered iff

$$2(t_1 + \cdots + t_i) + x \leq \lambda x$$

$$\Leftrightarrow \quad x \geq \frac{1}{\mu}(t_1 + \cdots + t_i),$$

$$\mu := \tfrac{\lambda - 1}{2}$$

Therefore, for a fixed robot and its two consecutive rounds, if it λ-covers intervals $[x_1, t_1]$ and $[x_2, t_2]$, then we can assume that $x_1 \leq x_2$.

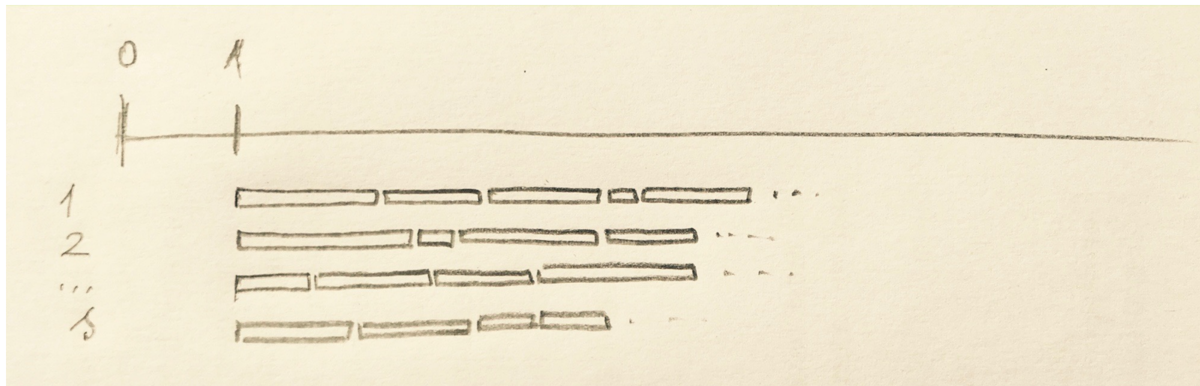
The intervals, λ-covered by robots in some rounds, must altogether form a q-fold covering of $\mathbb{R}_{\geq 1}$.

By moving some of the left endpoints of these intervals to the right, we can assume that each point of $\mathbb{R}_{\geq 1}$ is covered exactly q times.

Moreover, we can do it in a way that for each robot the left endpoints of the assigned intervals still form a non-decreasing sequence.

Denote $(x_i^{(r)}, t_i^{(r)}]$ the corresponding interval for the i-th round of robot r.

$\left(x_i^{(r)}, t_i^{(r)}\right]$, $i \in \mathbb{N}$, $r = 1, 2 \ldots, k$, cover $\mathbb{R}$ exactly q times.

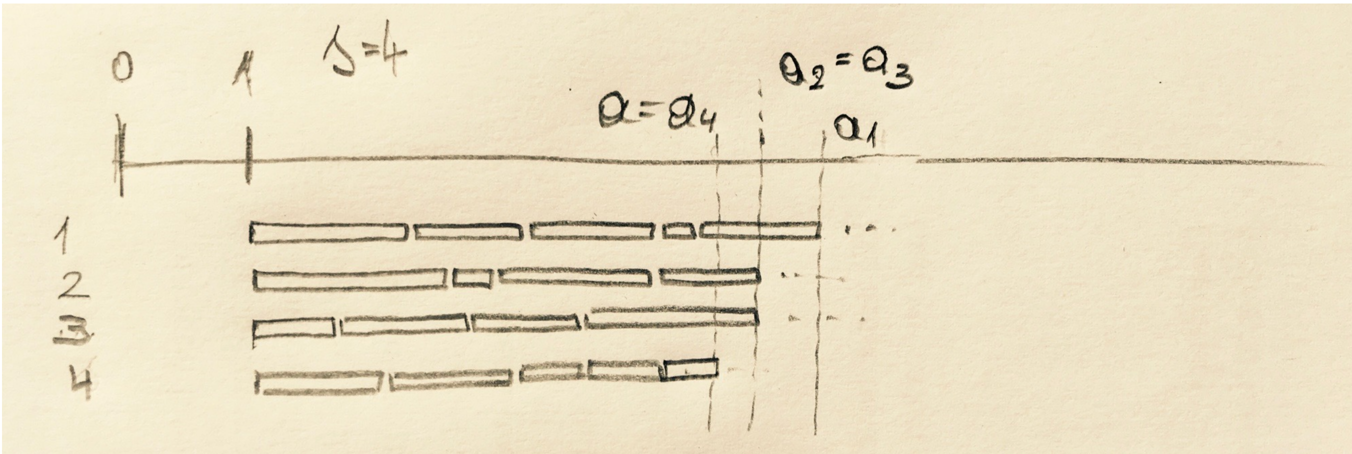


$$t_1 + \cdots + t_i \leq \mu x_{i+1}$$

$$\Leftrightarrow \quad t_i \leq \mu x_{i+1} - (t_1 + \cdots + t_{i-1})$$

Collect these intervals in a common sequence, sorted by left endpoints (ties broken arbitrarily).

Consider a prefix $\mathcal{P}$ of this sequence of intervals



$\mathcal{P}$ covers q times up to a point $a = a(\mathcal{P})$ and there are numbers

$$a = a_q \leq a_{q-1} \leq \cdots \leq a_1$$

such that $\mathcal{P}$ covers j times for $(a_{j+1}, a_j]$ and not at all for (a_1, ∞).

$$A(\mathcal{P}) := \{a_q, a_{q-1}, \ldots, a_1\}$$

a multiset describing the "current covering situation".

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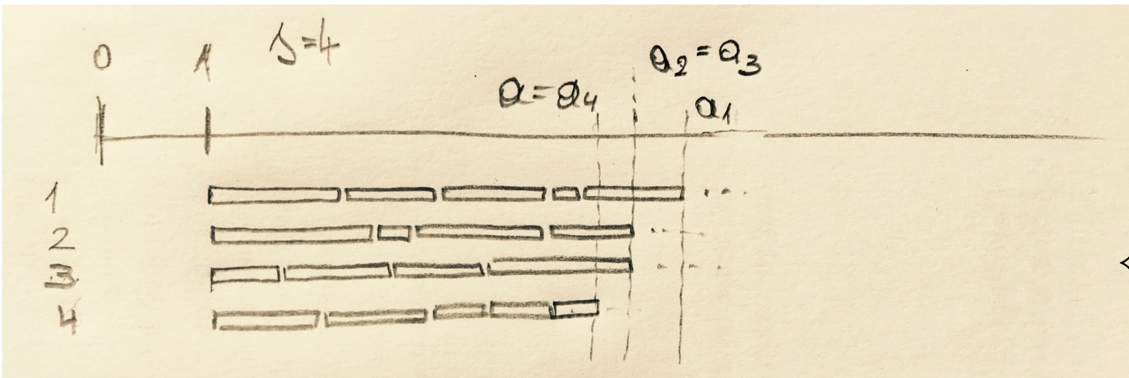

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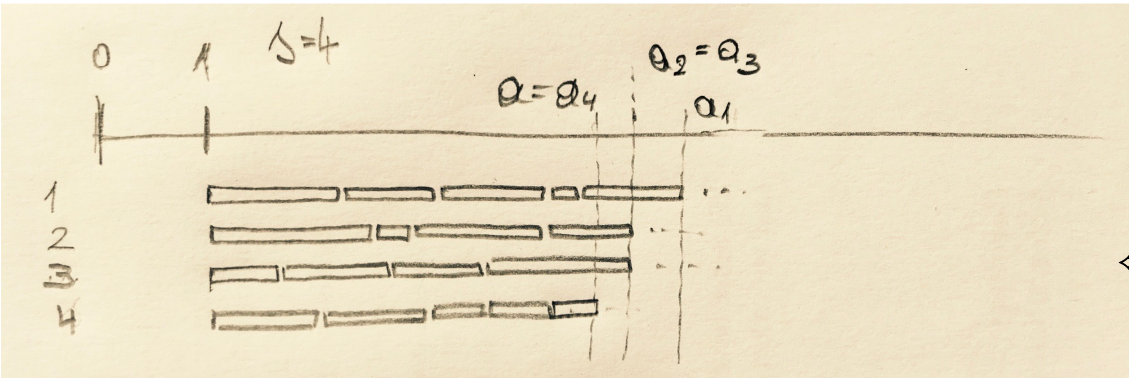

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The load of robot r in $\mathcal{P}$:

$L^{(r)}(\mathcal{P}) := t_1 + \ldots + t_{i_r}$, where $(x_{i_r}, t_{i_r}]$ is r's last interval in $\mathcal{P}$

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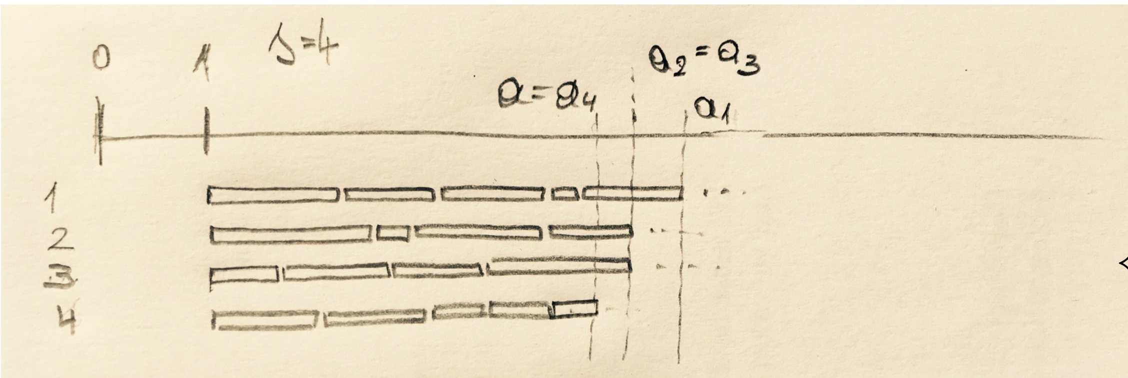

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Observe $\qquad L^{(r)}(\mathcal{P}) \leq \mu x_{i_r+1}$.

Intuition: Large a_i's and small $L^{(r)}(\mathcal{P})$'s are good for progress!

Compress status quo in a function

$$f(\mathcal{P}) := \prod_{r=1}^{k} \frac{\left(L^{(r)}(\mathcal{P})\right)^{q-k} (x^{(r)}_{i_r+1})^k}{\prod_{y \in A(\mathcal{P})} y}$$

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where $C = \max_{i,r} \frac{x^{(r)}_{i+1}}{x^{(r)}_{i}}$. If C is bounded, then $f(\mathcal{P})$ is bounded!

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$$\Rightarrow t_{i_j+1} = \mu^* x^{(j)}_{i_j+2} - L^{(j)}(\mathcal{P}),$$

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Put $b := x^{(j)}_{i_j+2}$. Then

$$\frac{f(\mathcal{P}^+)}{f(\mathcal{P})} = \frac{(\mu^* b)^{q-k} b^k}{\left(L^{(j)}(\mathcal{P})\right)^{q-k} \left(\mu^* b - L^{(j)}(\mathcal{P})\right)^k} = \frac{\mu^{*q-k}}{y^{q-k}(\mu^* - y)^k}$$

$$y := \frac{L^{(j)}(\mathcal{P})}{b}, \; 0 < \mu^* \le \mu$$

By simple high school calculus . . .

For $0 < x < \mu^* \leq \mu$,

$$\frac{{\mu^*}^{q-k}}{x^{q-k}(\mu^* - x)^k} \geq \frac{q^q}{(q-k)^{q-k}k^k{\mu^*}^k}$$

and thus

$$\frac{{\mu^*}^{q-k}}{x^{q-k}(\mu^* - x)^k} \geq \delta$$

for $\delta := \frac{q^q}{(q-k)^{q-k}k^k\mu^k} > 1$, provided $\mu < \sqrt[k]{\frac{q^q}{(q-k)^{q-k}k^k}} = \frac{\rho^\rho}{(\rho-1)^{\rho-1}}$.

For $\rho := \frac{q}{k}$. If $\delta > 1$, then $f(\mathcal{P})$ is unbounded — contradiction.

What if $C = \max_{i,r} \frac{x_{i+1}^{(r)}}{x_i^{(r)}}$ is unbounded? We do induction on k: a large interval is covered $q-1$ times by $k-1$ robots, and it has larger competitive ratio.

What if $C = \max_{i,r} \frac{x_{i+1}^{(r)}}{x_i^{(r)}}$ is unbounded? Then we do induction: basically, a large interval is covered $q-1$ times by $k-1$ robots, and it has larger competitive ratio.

Recall $\lambda = 2\mu + 1$.

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$$2\frac{\rho^\rho}{(\rho-1)^{\rho-1}} + 1. \tag{3}$$