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On the chromatic numbers of small-dimensional Euclidean spaces

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Abstract

In this paper, we will give a short survey concerning estimates on the chromatic numbers of Euclidean spaces in small dimensions. We will also present some new important bounds.

Keywords: Chromatic numbers, distance graphs.

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1 Introduction

This paper is concerned with the classical Nelson – Erdős – Hadwiger problem on painting the space $\mathbb{R}^n$ using the smallest possible number $\chi(\mathbb{R}^n)$ of colours in such a way that no two monochromatic points are at the distance 1 apart. In other words,

$$\chi(\mathbb{R}^n) = \min\left\{\chi : \ \mathbb{R}^n = V_1 \bigsqcup \ldots \bigsqcup V_\chi, \ \forall\, i \ \forall\, \mathbf{x}, \mathbf{y} \in V_i \ \ |\mathbf{x} - \mathbf{y}| \neq 1\right\}.$$

The quantity $\chi(\mathbb{R}^n)$ is called *the chromatic number of the space*.

Almost obviously, $\chi(\mathbb{R}^1) = 2$. However, even in the planar case, the problem still remains open. We only know that $4 \leq \chi(\mathbb{R}^2) \leq 7$ (see [8], [5]).

During the last decades, various upper and lower bounds have been obtained for the chromatic numbers of different spaces. In the next section, we will try to present the most recent results of this kind. In the last section, we will give our new improvements on those estimates.

2 An overview

First of all, we note that the problem we discuss here admits a natural reformulation in terms of graph theory. Indeed, we may consider the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where

$$\mathcal{V} = \mathbb{R}^n, \qquad \mathcal{E} = \{\{\mathbf{x}, \mathbf{y}\} : \ |\mathbf{x} - \mathbf{y}| = 1\}.$$

So the usual chromatic number $\chi(\mathcal{G})$ coincides with the value $\chi(\mathbb{R}^n)$. This interpretation suggests the study of colourings of *distance graphs*, i.e., of arbitrary subgraphs G in $\mathcal{G}$. All the best known lower bounds on $\chi(\mathbb{R}^n)$ are due to appropriate constructions of distance graphs.

For the three-dimensional space, the estimate $\chi(\mathbb{R}^3) \geq 5$ had been remaining unimproved for many years until in 2002 O. Nechushtan succeeded in replacing it by the bound $\chi(\mathbb{R}^3) \geq 6$ (see [9]). The tightest upper estimate was given in 2000 and published in 2002 by D. Coulson (see [4]): $\chi(\mathbb{R}^3) \leq 15$.

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In four dimensions, the bound $\chi(\mathbb{R}^4) \geq 7$ was independently proposed in 1996 by K. Cantwell (see [2]) and in 2006 by L. Ivanov (see [6]). The last construction is a bit simpler than the first one. The upper estimate $\chi(\mathbb{R}^4) \leq 49$ was announced by Coulson, but we could find in the litterature only the bound $\chi(\mathbb{R}^4) \leq 54$ (see [10]).

Starting from $n = 5$, we do not know any non-trivial upper estimates for the chromatic number. Certainly, $\chi(\mathbb{R}^n) \leq (\sqrt{n} + 1)^n$, which can be done by dividing the space into translates of a multicoloured cube. Also, we may cite D. Larman and C.A. Rogers' paper [7], where $\chi(\mathbb{R}^n) \leq (3 + o(1))^n$ is proved. However, nothing more precise and concrete has been done. So we just exhibit below a list of lower bounds for $5 \leq n \leq 12$:

- $\chi(\mathbb{R}^5) \geq 9$ (Cantwell, 1996, see [2]);
- $\chi(\mathbb{R}^6) \geq 11$ (J. Cibulka, 2008, see [3]);
- $\chi(\mathbb{R}^7) \geq 15$ (A.M. Raigorodskii, 2001, see [11]);
- $\chi(\mathbb{R}^8) \geq 16$ (see [12]);
- $\chi(\mathbb{R}^9) \geq 16$ (see [12]);
- $\chi(\mathbb{R}^{10}) \geq 19$ (Larman and Rogers, 1972, see [7]);
- $\chi(\mathbb{R}^{11}) \geq 20$ (Raigorodskii, 2001, see [11]);
- $\chi(\mathbb{R}^{12}) \geq 24$ (Larman and Rogers, 1972, see [7]).

As for growing dimensions, we have

$$(1.239 + o(1))^n \leq \chi(\mathbb{R}^n) \leq (3 + o(1))^n.$$

The lower bound is given in [11] and the upper one in [7].

For further references see [1], [11], [12].

3 New Results

First of all, we managed to achieve considerable improvements on lower bounds in the dimensions 9 – 12.

Theorem 1. *The estimates hold* $\chi(\mathbb{R}^9) \geq 21$, $\chi(\mathbb{R}^{10}) \geq 23$, $\chi(\mathbb{R}^{11}) \geq 23$, $\chi(\mathbb{R}^{12}) \geq 25$.

The results are obtained using a new series of distance graphs in the above-mentioned spaces.

At the same time, we propose a new general approach on "lifting" a bound in a dimension to a greater bound in a higher dimension:

Theorem 2. *Fix a natural* $n \geq 2$. *Suppose* G *is a distance graph which can be drawn on a sphere* $S \subset \mathbb{R}^n$ *whose radius* r_s *satisfies the inequalities* $\frac{1}{2} \leq r_s \leq \sqrt{\sqrt{3}-1}$ *and* $r_s \neq \sqrt{\frac{2}{3}}$. *Assume that* $\chi(G) \geq m$ *(so that* $\chi(\mathbb{R}^n) \geq m$*). Then* $\chi(\mathbb{R}^{n+2}) \geq m+4$.

Essentially, Theorem 2 sais that if $\chi(\mathbb{R}^n) \geq m$ and some special conditions are fulfilled, then $\chi(\mathbb{R}^{n+2}) \geq m+4$, i.e., enlarging the dimension by two, we enlarge the lower estimate by four.

Remark 1. *It is not hard to prove that, in the conditions of Theorem 2, if* $r_s = \sqrt{\frac{2}{3}}$, *then* $\chi(\mathbb{R}^{n+2}) \geq m+3$.

Remark 2. *Carefully applying Theorem 1 together with Theorem 2 we get* $\chi(\mathbb{R}^{11}) \geq 25$, *so eventually we have the following tabular:*

dim	1	2	3	4	5	6	7	8	9	10	11	12
$\chi \geq$	2	4	6	7	9	11	15	16	21	23	25	25

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