Families with forbidden subconfigurations

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Partitions

A subset $\mathcal{F} \subset 2^{[n]}$ is called a *family*.

A partition in $\mathcal{F}$: two disjoint sets $F_1, F_2 \in \mathcal{F}$, such that

$$F_1 \cup F_2 \in \mathcal{F}.$$

$$p(n) := \max\{|\mathcal{F}| : \mathcal{F} \subset 2^{[n]}, \mathcal{F} \text{ is partition-free}\}.$$

Partitions

For $n = 3m + i, \;\; i = 0, 1, 2,$ the family

$$\mathcal{K}(n) := \{K \subset [n] : m + 1 \leqslant |K| \leqslant 2m + 1\}$$

does not contain a partition.

Theorem (Kleitman, 1968)

$$p(3m+1) = |\mathcal{K}(3m+1)| = \sum_{m+1 \leqslant t \leqslant 2m+1} \binom{n}{t}.$$

Kleitman conjectured that $\boxed{p(n) = |\mathcal{K}(n)|}$ for any n.

Theorem 1 (P. Frankl, AK, 2017)

The conjecture is true. Moreover, we know all extremal families.

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Matchings

The matching number $\nu(\mathcal{F})$: the maximum number of pairwise disjoint members of $\mathcal{F}$.

$$e(n,s) := \max\left\{|\mathcal{F}| : \mathcal{F} \subset 2^{[n]}, \quad \nu(\mathcal{F}) < s\right\}.$$

$$e_k(n,s) := \max\left\{|\mathcal{F}| : \mathcal{F} \subset \binom{[n]}{k}, \ \nu(\mathcal{F}) < s\right\}.$$

Matchings

The case $s = 2$ corresponds to intersecting families.

Theorem (Erdős-Ko-Rado, 1938-1961)

$$e(n, 2) = 2^{n-1},$$

$$e_k(n, 2) = \binom{n-1}{k-1} \quad \text{for} \quad n \geqslant 2k.$$

Matchings. The non-uniform case

For $n < sm$ the family

$$\mathcal{B}(n,m) := \binom{[n]}{\geqslant m} := \{H \subset [n] : |H| \geqslant m\}$$

does not contain s pairwise disjoint sets.

Conjecture (Erdős, 1960's)

For $n = sm - 1$ we have $\quad e(n,s) = |\mathcal{B}(n,m)|$.

Theorem (Kleitman, 1966)

$$e(sm-1,s) = |\mathcal{B}(n,m)|,$$

$$e(sm,s) = \binom{sm-1}{m} + \sum_{m+1 \leqslant t \leqslant sm} \binom{sm}{t} \qquad (= 2e(sm-1,s)).$$

Matchings. The non-uniform case

Problem (Kleitman, 1966)

Determine $e(n, s)$ for other values of n.

Very little progress over 50 years...

Theorem (Quinn, 1987)

$$e(3m+1, 3) = \binom{3m}{m-1} + \sum_{m+1 \leqslant t \leqslant 3m+1} \binom{3m+1}{t}.$$

Unfortunately, it was not published in a refereed journal.

Construction

Let $n = sm + s - l$, $0 < l \leqslant s$.

$$\mathcal{P}(s,m,l) := \{P \subset 2^{[n]} : |P| + |P \cap [l-1]| \geqslant m+1\}.$$

$\nu(\mathcal{P}(s,m,l)) < s$: for disjoint $F_1, \ldots, F_s$ we have

$$sm + s - 1 \geqslant \sum_{i=1}^{s} |F_i| + |F_i \cap [l-1]| \geqslant sm + s.$$

Theorem 2 (P. Frankl, AK, 2016)

$e(sm + s - l, s) = |\mathcal{P}(s,m,l)|$ holds for

$$(2)\ l = 2,$$
$$(1)\ s \geqslant lm + 3l + 3.$$

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Matchings. The uniform case

How to construct a large family $\mathcal{A} \subset \binom{[n]}{k}$, satisfying $\nu(\mathcal{A}) < s$?

$$\mathcal{A}_1^{(k)}(n,s) := \left\{ A \in \binom{[n]}{k} : A \cap [s-1] \neq \emptyset \right\}, \quad \mathcal{A}_k^{(k)}(n,s) := \binom{[sk-1]}{k}.$$

Erdős Matching Conjecture, 1965

For $n \geqslant sk$ we have

$$e_k(n,s) = \max\{|\mathcal{A}_1^{(k)}(n,s)|, |\mathcal{A}_k^{(k)}(n,s)|\}.$$

True for $k \leqslant 3$ (Erdős and Gallai; Łuczak and Mieczkowska; Frankl).

$$e_k(n,s+1) = \binom{n}{k} - \binom{n-s+1}{k} \qquad \text{for} \quad n \geqslant (2s-1)k - s \quad \text{(Frankl)}.$$

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Stability Results

The covering number $\tau(\mathcal{H})$ of a family is the minimum of $|T|$ over all T satisfying $T \cap H \neq \emptyset$ for all $H \in \mathcal{H}$.

Hilton-Milner, 1967

Let $n \geqslant 2k$ and $\mathcal{F} \subset \binom{[n]}{k}$ satisfy $\nu(\mathcal{F}) < 2$ and $\tau(\mathcal{F}) \geqslant 2$. Then

$$|\mathcal{F}| \leqslant \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1 \qquad \text{holds.}$$

Stability Results

Theorem 4 (P. Frankl, AK, 2016)

Assume that $\nu(\mathcal{F}) < s,\ \tau(\mathcal{F}) \geqslant s$. Then the following holds:

$$|\mathcal{F}| \leqslant \binom{n}{k} - \binom{n-s+1}{k} - \binom{n-s+1-k}{k-1} + 1,$$

provided $k \geqslant 3,\ \ n \geqslant \big(2+o(1)\big)sk$, where $o(1)$ depends on s only.

Known to be true for $n > 2k^3 s$: Bollobás, Daykin and Erdős (1976).

Implications for other problems: anti-Ramsey type questions, non-uniform families with no large matchings, degree versions.

Open problems.

Let $\alpha_1 \geqslant \ldots \geqslant \alpha_n \geqslant 0$ be reals, $\sum_i \alpha_i < s$. Put $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$.

$$\mathcal{F}(\boldsymbol{\alpha}) := \{F \in 2^{[n]} : \sum_{i \in F} \alpha_i \geqslant 1\}.$$

Then $\nu(\mathcal{F}(\boldsymbol{\alpha})) < s$ holds. Also $\mathcal{F}(\boldsymbol{\alpha}) = \{0,1\}^n \cap \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x}, \boldsymbol{\alpha} \rangle \geqslant 1\}$.

Conjecture (P. Frankl, AK, 2016)

For any n, s the maximum of $e(n,s)$ (or $e_k(n,s)$) is attained on the family $\mathcal{F}(\boldsymbol{\alpha})$ for suitable $\boldsymbol{\alpha} \in \mathbb{R}^n$.

Non-uniform case:

Maximum size of families without certain structures involving disjoint sets (e.g., r-partition free families, introduced by Frankl in 1977).