Embedding graphs in Euclidean space

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Abstract

The dimension of a graph G is the smallest d for which it can be embedded in $\mathbb{R}^d$ as a unit distance graph. Answering a question of Erdős and Simonovits, we show that any graph with less than $\binom{d+2}{2}$ edges has dimension at most d. Improving their result, we also prove that the dimension of a graph with maximum degree d is at most d.

Keywords: Unit distance graph, Graph representation, Graph dimension

1 Introduction

Definition 1.1 A graph $G = (V, E)$ is a *unit distance graph* in $\mathbb{R}^d$, if $V \subset \mathbb{R}^d$ and

$$E \subseteq \{(x, y) : x, y \in V, |x - y| = 1\}.$$

Note that we do not require the edge set of a unit distance graph to contain all unit-distance pairs.

We say that a graph G is *realizable* in a subset X of $\mathbb{R}^d$, if there exists a unit distance graph G' in $\mathbb{R}^d$ on a set of vertices $X_0 \subset X$, which is isomorphic to G. We will use this notion for $X = \mathbb{R}^d$ and for $X = \frac{1}{\sqrt{2}}\mathbb{S}^{d-1}$, where $\frac{1}{\sqrt{2}}\mathbb{S}^{d-1}$ is a sphere of radius $1/\sqrt{2}$ with center in the origin.

In the paper [3], Erdős, Harary and Tutte introduced the concept of the Euclidean dimension $\dim G$ of a graph G.

Definition 1.2 The *Euclidean dimension* $\dim G$ (*spherical dimension* $\dim_S G$) of a graph G is equal to k, if k is the smallest integer such that G is realizable in $\mathbb{R}^k$ (on $\frac{1}{\sqrt{2}}\mathbb{S}^{k-1} \subset \mathbb{R}^k$).

They studied the dimension of graphs, e.g, complete graphs, wheels, complete bipartite graphs, cubes. They also study the relation of the dimension to the chromatic number of the graph and to its girth.

In [4] it was shown that if G has maximum degree d then $\dim G \leq \dim_S G \leq d + 2$. We prove something stronger.

Theorem 1.3 *Let $d \geq 2$. Any graph $G = (V, E)$ with maximum degree $d - 1$ has spherical dimension at most d.*

We also prove the following.

Theorem 1.4 *Let $d \geq 1$ and let $G = (V, E)$ be a graph with maximum degree d. Then G is a unit distance graph in $\mathbb{R}^d$ except if $d = 3$ and G contains $K_{3,3}$.*

Definition 1.5 Let $f(d)$ denote the least number for which there is a graph with $f(d)$ edges that is not realizable in $\mathbb{R}^d$.

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[1] Research was partially supported by the National Research, Development, and Innovation Office, NKFIH Grant K119670

[2] Research supported in part by Swiss National Science Foundation grants no. 200020-162884 and 200021-175977 and by the grant N 15-01-03530 of the Russian Foundation for Basic Research

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There are some natural upper bounds on $f(d)$. Since K_{d+2} is not realizable in $\mathbb{R}^d$, it is clear that $f(d) \le \binom{d+2}{2}$. For $d = 3$ we have $f(d) \le \binom{d+2}{2} - 1$, because $K_{3,3}$ can not be embedded in $\mathbb{R}^3$. Speaking of lower bounds House [5] proved that $f(3) = 9$, and that $K_{3,3}$ is the only graph with 9 edges that can not be realized in $\mathbb{R}^3$. Chaffee and Noble [2] showed that $f(4) = \binom{4+2}{2} = 15$, and there are only two graphs, K_6 and $K_{3,3,1}$, with 15 edges that can not be realized in $\mathbb{R}^4$ as a unit distance graph.

In [4], Erdős and Simonovits asked if $f(d) = \binom{d+2}{2}$ for $d > 3$. We confirm this below.

Theorem 1.6 *Let $d > 3$. Any graph G with less than $\binom{d+2}{2}$ edges can be embedded in $\mathbb{R}^d$. Moreover, if the graph does not contain $K_{d+2} - K_3$ or K_{d+1}, it can be embedded on $\frac{1}{\sqrt{2}}\mathbb{S}^{d-1}$.*

Ramsey-type questions about unit distance graphs have been studied by Kupavskii, Raigorodskii and Titova in [6] and by Alon and Kupavskii in [1]. In [1] the authors introduced the quantity $f_D(s)$, which is the smallest possible d, such that for any graph G on s vertices either G or its complement $\overline{G}$ can be realized as a unit distance graph in $\mathbb{R}^d$, and proved that $f_D(s) = (\frac{1}{2} + o(1))s$. Similarly we define $f_{SD}(s)$ to be the smallest possible d, such that for any graph G on s vertices either G or its complement $\overline{G}$ can be realized as a unit distance graph on $\frac{1}{\sqrt{2}}\mathbb{S}^{d-1}$. We determine the exact value of $f_{SD}(s)$ and give almost sharp bounds on $f_D(s)$.

Theorem 1.7 *$f_{SD}(s) = \lceil (s+1)/2 \rceil$ and $\lceil (s-1)/2 \rceil \le f_D(s) \le \lceil (s+1)/2 \rceil$.*

2 Maximum degree

In the proofs of the bounded maximum degree results we use the following lemma of Lovász.

Lemma 2.1 ([7]) *Let $G = (V, E)$ be a graph with maximum degree k and $k_1, \dots, k_\alpha$ be non-negative integers such that $k_1 + \cdots + k_\alpha = k - \alpha + 1$. Then there is a partition $V = V_1 \cup \cdots \cup V_\alpha$ of the vertex set into α parts such that the maximum degree in $G[V_i]$ is at most k_i, $i = 1, \dots, \alpha$.*

We apply this lemma for $\alpha = 2$, to prove Theorem 1.3.

Proof of Theorem 1.3 The proof is by induction on d. For $d = 2$ and $d = 3$ the theorem is easy to verify. Let $V = V_1 \cup V_2$ be a partition as in Lemma 2.1 for $k_1 = \lfloor \frac{d-2}{2} \rfloor$, $k_2 = \lceil \frac{d-2}{2} \rceil$. Then by the induction hypothesis, $G[V_i]$ can be represented on $\frac{1}{\sqrt{2}}\mathbb{S}^{k_i}$. Represent $G[V_1]$ and $G[V_2]$ on the spheres of radius $\frac{1}{\sqrt{2}}$

centered at the origin in orthogonal subspaces of dimension k_1+1 and k_2+1. Both spheres are subspheres of $\frac{1}{\sqrt{2}}\mathbb{S}^d$. □

In the proof of Theorem 1.4 we use the following lemma and proposition. The lemma is a strengthening of a special case of Lemma 2.1.

Lemma 2.2 *Let $G=(V,E)$ be a graph with maximum degree d.*

If d is even, then there is a partition $V = V_1 \cup \cdots \cup V_{d/2}$ such that the maximum degree of $G[V_i]$ is at most 1 if $1 \le i < d/2$, the maximum degree of $G[V_{d/2}]$ is at most 2, and any $v \in V_{d/2}$ of degree 2 in $G[V_{d/2}]$ has exactly 2 neighbours in each V_i.

If d is odd, then there is a partition $V = V_1 \cup \cdots \cup V_{(d-1)/2}$ such that the maximum degree of $G[V_i]$ is at most 1 if $1 \le i < (d-3)/2$, the maximum degree of $G[V_{(d-3)/2}]$ and $G[V_{(d-1)/2}]$ is at most 2, and any $v \in V_{(d-3)/2}$ of degree 2 in $G[V_{(d-3)/2}]$ has exactly 2 neighbours in each V_i for $i \le (d-3)/2$ and exactly 3 neighbours in $V_{(d-1)/2}$.

The following proposition states that paths and cycles can be embedded on the sphere of radius $\frac{1}{\sqrt{2}}$ in $\mathbb{R}^3$ in a sufficiently general position. Note that when a 4-cycle is embedded on $\frac{1}{\sqrt{2}}\mathbb{S}^2$, there is always a pair non-adjacent points that are in opposite positions on the sphere.

Proposition 2.3 *Any graph $G=(V,E)$ with maximum degree 2 can be embedded on a sphere of radius $\frac{1}{\sqrt{2}}$ in $\mathbb{R}^3$ such that the following hold:*

(i) *For any 3 distinct vertices a, b, c no vertex is at distance 1 from a, b and c.*

(ii) *No 4 vertices are on a circle, except for those 4-tuples that are formed by two opposite-position pairs of two 4-cycles.*

In the proof we use ideas from the correction [9] to the paper [8] of Lovsz, Saks and Schrijver. For a graph $G=(V,E)$ let $v_1,\dots,v_n$ be an ordering of the vertices. for each v_i choose vectors u_i of length $\frac{1}{\sqrt{2}}$ independently uniformly at random in $\mathbb{R}^d$. We modify the vectors u_i to obtain a unit distance representation of G by an orthogonalization process. For each i from 2 to n we project u_i in the orthogonal complement of the subspace spanned by

$$L_i = \{u_j \,:\, j < i \text{ and } (v_i v_j) \in E\}\,.$$

With this method for every ordering of the vertices we have a probability distribution on the spherical unit distance representations of G. The distributions that correspond to different orderings may be different, but the following is true.

Lemma 2.4 **([9])** *For any graph $G = (V, E)$ if G does not contain a complete bipartite graph on $d+1$ vertices, the distributions, given by random realizations, in $\mathbb{R}^d$ for any two ordering of the vertices have the same sets of measure zero.*

Sketch of proof of Proposition 2.3 G is a disjoint union of paths and cycles. If we remove a vertex from each 4-cycle, we obtain a graph G' on the vertex set $V' = v_1, \ldots, v_n$ that does not contain a complete bipartite graph on 4 vertices (that is, it does not contain a 4-cycle). Take a random realization of G' as described above, and then add back the removed vertices as follows. If A was removed from the cycle $ABCD$ with this cyclic order, then embed A as the point opposite to C. We claim that with probability 1 this realization satisfies the conditions of the proposition. □

Proof of Theorem 1.4 For $d = 1$ and $d = 2$, the statement is trivial, and for $d = 3$, it follows from Proposition 2.3. First we remove vertices of degree 3 in G from V one by one. Let $W \subset V$ be the set of removed vertices. Each $w \in W$ has exactly 3 neighbours in V, W is an independent set of G, and the maximum degree in $G[V \setminus W]$ is at most 2. Now we represent $G[V \setminus W]$ on a 2-sphere of radius $\frac{1}{\sqrt{2}}$ as in Proposition 2.3. Finally, we embed the removed vertices in W one by one as follows. For any circle on the sphere there are exactly 2 points at distance 1 from the circle. (They are not necessarily on the sphere.) If $w_1 \in W$ and $w_2 \in W$ have different sets of neighbours, then their neighbours span different circles on the sphere. (This is because if the set of neighbours of $w \in W$ span a circle that contains 4 vertices, then all of these 4 vertices are from 4-cycles, hence have degree 2 in $G[V \setminus W]$. So no 2 vertices in W are joined to 3 vertices on this circle.) Moreover, there are no 3 vertices in W with the same set of neighbours, because G does not have $K_{3,3}$ as a component.

For $d > 3$ we consider two cases depending on the parity of d. We only present the proof for the odd case here, for the even it is similar.

Assume that d is odd. Let $V = V_1 \cup \cdots \cup V_{(d-3)/2} \cup V_{(d-1)/2}$ be a partition as in Lemma 2.2, and as in Case 1, find an independent set $W \subseteq V_{(d-3)/2}$ of G such that each $w \in W$ has exactly 2 neighbours in $V_{(d-3)/2}$ and the maximum degree in $G[V_{(d-3)/2} \setminus W]$ is at most 1. Then we can embed $G[V \setminus W]$ on $(d-3)/2$ circles and a 2-sphere such that the circles and the 2-sphere span pairwise orthogonal subspaces. Embed each $G[V_i]$ $(1 \leq i < (d-3)/2)$ and $G[V_{(d-3)/2} \setminus W]$ on circles of radius $\frac{1}{\sqrt{2}}$ in orthogonal planes, with no two vertices opposite, and embed $V_{(d-1)/2}$ on a 2-sphere as in Proposition 2.3. Note that the embedded $V \setminus W$ lies on the sphere S of radius $\frac{1}{\sqrt{2}}$ with the center in the origin.

Then we can add the vertices of W one by one to this embedding. Each $w \in W$ has exactly 2 neighbours in each V_i for $1 \leq i \leq (d-3)/2$ and 3 neighbours in $V_{(d-1)/2}$. The set of neighbours $N(w)$ spans an affine subspace A of dimension $d-1$. If A does not pass through the origin, then the 2 points at distance 1 from $N(w)$ are not on the sphere S. If A passes through the origin, the 2 points at distance 1 from $N(w)$ lie on the 2-sphere on which $G[V_{(d-1)/2}]$ is embedded, but since $G[V_{(d-1)/2}]$ is embedded on the 2-sphere such that no vertex is at the pole of the circle through any 3 vertices, the 2 possible positions for w do not coincide with the position of any vertex in $V \setminus W$. Finally, for any $w_1 \in W$ and $w_2 \in W$ if $N(w_1) \neq N(w_2)$, then $N(w_1)$ and $N(w_2)$ span different affine subspaces, and there are no 3 vertices in W with the same neighbours, because the maximum degree of $G[V_{(d-3)/2}]$ is at most 2. □

3 Number of edges

A graph $G = (V, E)$ is called *k-degenerate* if any subgraph of G has a vertex of degree at most k. The *colouring number* of G is the least k for which there exists an ordering $v_1, v_2, \ldots, v_n$ of the vertices in which each vertex has less than k neighbors of smaller index. It is not difficult to see that the colouring number of G is at most k if and only if it is k-degenerate.

Lemma 3.1 *Let $d \geq 2$ and x be a vertex of degree at most $d-2$ in a graph G. If $G - x$ can be realized on $\frac{1}{\sqrt{2}}\mathbb{S}^{d-1}$ as a unit distance graph, then G can also be represented on $\frac{1}{\sqrt{2}}\mathbb{S}^{d-1}$.*

Proof. The neighbours of x span a linear subspace of dimension at most $d-2$, so there is a great circle from which to choose x. □

The following corollary also follows from the proof of Proposition 2 in [4].

Corollary 3.2 *Any graph with colouring number at most $d-2$ has spherical dimension at most d.*

Proof of Theorem 1.6 Let $G = (V, E)$ a graph with less than $\binom{d+2}{2}$ edges. If G is $(d-2)$-degenerate, then by Corollary 3.2 it can be embedded on a sphere, so we may assume that G is not $(d-2)$-degenerate. Let H be a maximal subgraph of G of minimum degree at least $d-1$. For the number of vertices m of H we have $d \leq m \leq d+5$, because if $m \geq d+6$, then the number of edges of H is at least $\frac{(d+5)(d-1)}{2} > \binom{d+2}{2} - 1$ if $d \geq 4$.

There are 6 possible values of m to consider, $m = d, d+1, d+2, d+3, d+4, d+5$. The rest of the proof consists of analyzing these cases.

□

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