Families with no matchings of size s

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Abstract

Let $k \geqslant 2$, $s \geqslant 2$ be positive integers. Let $[n]$ be an n-element set, $n \geqslant ks$. Subsets of $2^{[n]}$ are called *families*. If $\mathcal{F} \subset \binom{[n]}{k}$, then it is called k-*uniform*. What is the maximum size $e_k(n,s)$ of a k-uniform family without s pairwise disjoint members? The well-known Erdős Matching Conjecture would provide the answer for all n, k, s in the above range. For $n > 2ks$ it is known that the maximum is attained by $\mathcal{A}_1(T) := \{A \subset [n] : |A| = k, A \cap T \neq \emptyset\}$ for some fixed $(s-1)$-element set $T \subset X$. We discuss recent progress on this problem. In particular, our recent stability result states that for $n > (2+o(1))ks$ and a k-uniform family $\mathcal{F}$, $\mathcal{F} \nsubseteq \mathcal{A}_1(T)$, then $|\mathcal{F}|$ is considerably smaller.

This result is applied to obtain the corresponding anti-Ramsey numbers in a wide range.

Removing the condition of uniformness, we arrive at another classical problem of Erdős, which was solved by Kleitman for $n \equiv 0$ or $-1 \pmod{s}$. We succeeded in resolving this long-standing problem for $n \equiv -2 \pmod{s}$ via a new averaging technique which might prove useful in various other situations.

1 Introduction

Put $[n] := \{1, 2, \ldots, n\}$ and let $2^{[n]}$ denote the power set of $[n]$. A subset $\mathcal{F} \subset 2^{[n]}$ is called a *family of subsets of* $[n]$, or simply a *family*. For $0 \leqslant k \leqslant n$ we use the notation $\binom{[n]}{k} := \{H \subset [n] : |H| = k\}$.

The maximum number of pairwise disjoint members of a family $\mathcal{F}$ is denoted by $\nu(\mathcal{F})$ and is called the *matching number* of $\mathcal{F}$. Let us define the following two quantities for $n, k, s \geqslant 2$:

$$e(n,s) := \max\left\{|\mathcal{F}| : \mathcal{F} \subset 2^{[n]}, \nu(\mathcal{F}) < s\right\},$$
$$e_k(n,s) := \max\left\{|\mathcal{F}| : \mathcal{F} \subset \binom{[n]}{k}, \nu(\mathcal{F}) < s\right\}.$$

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For $s = 2$ both quantities were determined by Erdős, Ko and Rado [4]: $e(n,2) = 2^{n-1}$, $e_k(n,2) = \binom{n-1}{k-1}$ for $n \geqslant 2k$. For $s \geqslant 3$ this becomes a much harder task. Let us first discuss the k-uniform problem.

2 The uniform case. Stability

The following families are the natural candidates for being an extremal family of k-sets with no $(s+1)$-matching:

$$\mathcal{A}_i^{(k)}(n,s) := \Big\{A \in \binom{[n]}{k} : \big|A \cap [(s+1)i - 1]\big| \geqslant i\Big\}, \quad 1 \leqslant i \leqslant k. \tag{1}$$

Conjecture 1 (Erdős Matching Conjecture [2]). *For $n \geqslant (s+1)k$ we have*

$$e_k(n, s+1) = \max\big\{|\mathcal{A}_1^{(k)}(n,s)|, |\mathcal{A}_k^{(k)}(n,s)|\big\}. \tag{2}$$

Conjecture 1 is known to be true for $k \leqslant 3$ (cf. [3], [12] and [6]). Improving some earlier results, in [5] it is shown that

$$e_k(n, s+1) = |\mathcal{A}_1^{(k)}(n,s)| = \binom{n}{k} - \binom{n-s}{k} \qquad \text{for} \quad n \geqslant (2s+1)k - s. \tag{3}$$

Although we did not progress on the conjecture, we advanced in understanding the structure of families with no s-matchings that have cardinalities close to $e_k(n,s)$. Let us define the *covering number* $\tau(\mathcal{F})$: $\tau(\mathcal{F}) := \min\{|C| : C \subset [n], \forall F \in \mathcal{F} \;\; C \cap F \neq \emptyset\}$. Note that $\tau(\mathcal{A}_1^{(k)}(n,s)) = \nu(\mathcal{A}_1^{(k)}(n,s)) = s$ for $n \geqslant (s+1)k$. In the case $s = 2$ one has a very useful stability theorem due to Hilton and Milner [10]. Below we discuss this theorem together with its natural generalization to the case $s \geqslant 2$. Put $X := [s+1, s+k]$ and consider the following family:

$$\mathcal{H}^{(k)}(n,s) := X \cup \Big\{H \in \binom{[n]}{k} : H \cap [s-1] \neq \emptyset \;\text{ or }\; s \in H, H \cap X \neq \emptyset\Big\}.$$

Note that $\nu(\mathcal{H}^{(k)}(n,s)) = s < \tau(\mathcal{H}^{(k)}(n,s))$ for $n \geqslant sk$ and

$$|\mathcal{H}^{(k)}(n,s)| = \binom{n}{k} - \binom{n-s}{k} + 1 - \binom{n-s-k}{k-1}. \tag{4}$$

Theorem (Hilton-Milner [10]). *Suppose that $n \geqslant 2k$ and let $\mathcal{F} \subset \binom{[n]}{k}$ be a family satisfying $\nu(\mathcal{F}) = 1$ and $\tau(\mathcal{F}) \geqslant 2$. Then*

$$|\mathcal{F}| \leqslant |\mathcal{H}^{(k)}(n,1)| \qquad \textit{holds.}$$

The following generalization of the Hilton-Milner theorem for $s \geqslant 2$ is obtained in [8].

Theorem 1 (Frankl, Kupavskii [8]). *Suppose that $k \geqslant 3, n \geqslant \big(2 + o(1)\big)sk$, where $o(1) \to 0$ as $s \to \infty$. Then for any $\mathcal{G} \subset \binom{[n]}{k}$ with $\nu(\mathcal{G}) = s < \tau(\mathcal{G})$ we have $|\mathcal{G}| \leqslant |\mathcal{H}^{(k)}(n,s)|$.*

Informally, it states that any family of size larger than $|\mathcal{H}^{(k)}(n,s)|$ must be a subfamily of $\mathcal{A}_1^{(k)}(n,s)$. We mention that for $n \geqslant 2k^3 s$ this theorem was proven by Bollobás, Daykin and Erdős [1].

3 An application to an anti-Ramsey problem

Theorem 1 seems to be useful to attack other hypergraph problems. In [8] we apply a similar result to the following anti-Ramsey problem. Consider a coloring of $\binom{[n]}{k}$ with M colors (each color must be present in the coloring). The *anti-Ramsey number* $ar(n,k,s)$ is the minimum M such that in any such coloring there is a *rainbow* s-matching, that is, a set of s pairwise disjoint k-sets from pairwise distinct color classes.

This quantity was studied by Özkahya and Young [13], who have made the following conjecture.

Conjecture 2 ([13])**.** *One has* $ar(n,k,s) = e_k(n,s-1)+2$ *for all* $n > sk$.

It is not difficult to see that $ar(n,k,s) \geqslant e_k(n,s-1)+2$ for any n,k,s. Indeed, consider the largest family of k-sets with no $(s-1)$-matching and assign a different color to each of its sets. Next, assign one new color to all the remaining sets. This is a coloring of $\binom{[n]}{k}$ in $e_k(n,s-1)+1$ colors without a rainbow s-matching.

In [13] the authors proved this conjecture for $s=3$ and for $n \geqslant 2k^3 s$. They also obtained the bound $ar(n,k,s) \leqslant e_k(n,s-1)+s$ for $n \geqslant sk+(s-1)(k-1)$. In [8] we prove the following theorem.

Theorem 2 (Frankl, Kupavskii [8])**.** *We have* $ar(n,k,s) = e_k(n,s-1)+2$ *for* $n \geqslant sk+(s-1)(k-1)$, $k \geqslant 3$.

We remark that the case $k=2$ of Conjecture 2 has already been settled for all values of parameters (see [13] for the history of the problem). Theorem 2 is actually a consequence of a much stronger result proven in [8], we refer to [8] for details.

4 The non-uniform case

Let us discuss the quantity $e(n,s)$ in this section. What are the families in $2^{[n]}$ with no s-matchings? One natural example of such family is $\binom{[n]}{\geqslant m}$ for $m = \lceil \frac{n+1}{s} \rceil$. Erdős conjectured that for $n = sm-1$ this family is extremal. Half a century ago Kleitman proved this conjecture and also determined $e(sm,s)$.

Theorem (Kleitman [11])**.**

$$e(sm-1,s) = \sum_{m \leqslant t \leqslant sm-1} \binom{sm-1}{t}, \tag{5}$$

$$e(sm,s) = \binom{sm-1}{m} + \sum_{m+1 \leqslant t \leqslant sm} \binom{sm}{t}. \tag{6}$$

For $s = 2$ both formulae give 2^{n-1}, the easy-to-prove bound from the Erdős-Ko-Rado theorem. In the case $s = 3$ there is just one case not covered by the Kleitman Theorem, namely $n \equiv 1 \pmod 3$. This was the subject of the PhD dissertation of Quinn [14]. There he gave a very long and tedious proof of the following equality:

$$e(3m+1, 3) = \binom{3m}{m-1} + \sum_{m+1 \leqslant t \leqslant 3m+1} \binom{3m+1}{t}. \tag{7}$$

Unfortunately, this result was never published and no further progress was made on the problem until recently. In [7] and [9] we determined $e(n, s)$ for a relatively wide range of parameters. For the sake of brevity, we state the second part of the next theorem somewhat imprecisely.

Theorem 3 (Frankl, Kupavskii, [7], [9]). **1.** *Suppose that* $n = sm + s - 2$. *Then*

$$e(n, s) = \binom{n-1}{m-1} + \sum_{m+1 \leqslant t \leqslant n} \binom{n}{t}. \tag{8}$$

2. *Determination of* $e(sm + s - l, s)$ *for all* $s \geqslant lm + 3l + 3$.

To prove the first part of Theorem 3, we used two different methods for $s \leqslant 4$ and $s \geqslant 5$. Below we sketch the idea of the proof in the case $s = 3$, which provides us with a relatively short proof of Quinn's result [14]. We think that the proof method may be useful for other problems that ask for the largest families in $2^{[n]}$ without a certain configuration.

Idea of the proof for $n = 3m + 1$, $s = 3$. The proof is by an averaging argument. We start with a family $\mathcal{F}$ with no 3-matching. We assume that $\mathcal{F}$ is monotone: if it contains a set, then it contains all its supersets. Then we fix a permutation σ and a family $\mathcal{H}(\sigma)$, which we call a *test configuration*. The family $\mathcal{H}(\sigma)$ contains all $(\leqslant m)$-sets that form intervals in σ. Moreover, for each triple of pairwise disjoint m-sets, we take several $(m+i)$-sets, $i = 1, 2, 3$, which contain one of the m-sets and are disjoint with pairs of some of the $(\leqslant m)$-sets, creating new pairwise disjoint triples in $\mathcal{H}(\sigma)$. For a precise description of $\mathcal{H}(\sigma)$, see [9]. Its choice is essential for the proof. One important feature is that $\mathcal{H}(\sigma)$ splits into n subfamilies, each of which is very similar to a chain with respect to containment, each passing through exactly one m-set.

The next step is to analyze $\mathcal{F} \cap \mathcal{H}(\sigma)$. The family $\mathcal{H}(\sigma)$ is constructed in such a way that $\mathcal{F}$ intersects it in just a "right" amount of sets. To formalize it, we introduce charges and use a discharging method. We give charges to each set in $\mathcal{F} \cap \mathcal{H}(\sigma)$, and then redistribute them between the sets in $\mathcal{H}(\sigma)$ in order to show that the charge of $\mathcal{F} \cap \mathcal{H}(\sigma)$ is not too large. In the redistribution part we transfer the weight from the $(\leqslant m)$-sets to the $(\geqslant m+1)$-sets. Once completed, we average over σ to show that $\mathcal{F}$ cannot be larger than the bound from the theorem. We put different charges on the sets in $\mathcal{F} \cap \mathcal{H}(\sigma)$ to compensate for the fact that the proportion of i-sets contained in $\mathcal{H}(\sigma)$ is different for different i, which shows up when we do the averaging. □

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