Intersecting families

Andrey Kupavskii

Ecole Polytechnique Fédérale de Lausanne

A subset $\mathcal{F} \subset 2^{[n]}$ is called a *family*.

A family is intersecting, if any two of its sets intersect.

Theorem (Erdős-Ko-Rado, 1961)

Given $n \geqslant 2k > 0$, if a family $\mathcal{F}$ of k-subsets of $[n]$ is intersecting, then $|\mathcal{F}| \leqslant \binom{n-1}{k-1}$.

The bound in theorem is attained on a family of all k-sets containing a given element.

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Non-trivial intersecting families

Theorem (Hilton-Milner, 1967)

Given $n > 2k > 0$, if a family $\mathcal{F}$ of k-subsets of $[n]$ is intersecting and non-trivial, then

$$|\mathcal{F}| \leqslant \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

The diversity $\gamma(\mathcal{F})$ of a family $\mathcal{F}$ is the number of sets not containing the most popular element.

Theorem (Frankl, 1987)

Given $n > 2k > 0$ and an integer $3 \leqslant i \leqslant k$, if a family $\mathcal{F}$ of k-subsets of $[n]$ is intersecting and $\gamma(\mathcal{F}) \geqslant \binom{n-i-1}{n-k-1}$, then

$$|\mathcal{F}| \leqslant \binom{n-1}{k-1} - \binom{n-i-1}{k-1} + \binom{n-i-1}{n-k-1}.$$

The bound attained on a family, containing all k-sets containing $[2, i+1]$, and all k-sets containing 1 and intersecting $[2, i+1]$.

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Proof ingredients

A pair of families $\mathcal{A}, \mathcal{B}$ are called cross-intersecting if any set from $\mathcal{A}$ intersects all sets from $\mathcal{B}$.

A lexicographical order (lex) on $\binom{[n]}{k}$: A is before B iff the minimal element of $A \setminus B$ is less than the minimal element of $B \setminus A$.

For $0 \leqslant m \leqslant \binom{n}{k}$ let $\mathcal{L}(m, k)$ be the collection of first m k-sets with respect to lex.

Theorem (Kruskal 1963, Katona 1968)

Suppose that $\mathcal{A} \subset \binom{[n]}{a}, \mathcal{B} \subset \binom{[n]}{b}$ are cross-intersecting. Then the families $\mathcal{L}(|\mathcal{A}|, a), \mathcal{L}(|\mathcal{B}|, b)$ are also cross-intersecting.

Intersecting family $\rightarrow$ a pair of cross-intersecting families: the sets containing 1 and the sets not containing 1. Replace by lex families.

Difficult case: when $\gamma(\mathcal{F}) > \binom{n-3}{k-2}$.

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A stronger version of Frankl's theorem with a simpler proof:

Theorem (Kupavskii, Zakharov, 2016+)

Given $n > 2k > 0$ and a real $3 \leqslant u \leqslant k$, if a family $\mathcal{F}$ of k-subsets of $[n]$ is intersecting and $\gamma(\mathcal{F}) \geqslant \binom{n-u-1}{n-k-1}$, then

$$|\mathcal{F}| \leqslant \binom{n-1}{k-1} - \binom{n-u-1}{k-1} + \binom{n-u-1}{n-k-1}.$$

The main ingredient of the proof is Kruskal-Katona theorem, and an easy statement that in a regular bipartite graph the largest independent set is one of its parts.

We modify the family step by step, not decreasing its size and decreasing its diversity, until we arrive at the family giving equality in the theorem.

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Number of intersecting families

Theorem (Balogh, Das, Delcourt, Liu, and Sharifzadeh, 2015)
For $n \geqslant 3k + 8\log k$ and $k \to \infty$, most intersecting families are trivial.

They posed a problem to extend their result for smaller $n = n(k)$.

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For $n \geqslant 2k + 2 + 2\sqrt{k \log k}$ and $k \to \infty$, most intersecting families are trivial, and most non-trivial intersecting families have diversity 1.

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Degree versions

The degree of an element is the number of sets from the family containing it.

Theorem (Huang, Zhao, 2017)

Let $n \geqslant 2k+1 > 1$. Then any intersecting family has minimum degree at most $\binom{n-2}{k-2}$.

The proof is based on the application of the eigenvalue methods. They asked for a purely combinatorial proof of the theorem.

Frankl and Tokushige gave a combinatorial proof for $n \geqslant 3k$.

The degree of a subset $S \subset [n]$ is the number of sets from the family containing S. $\delta_t(\mathcal{F})$ is the minimal degree of an t-subset $S \subset [n]$.

Theorem (Kupavskii, 2017+)

If $n \geqslant 2k+2 > 2$, then for any intersecting family $\mathcal{F}$ of k-subsets of $[n]$ we have $\delta_1(\mathcal{F}) \leqslant \binom{n-2}{k-2}$. More generally, if $n \geqslant 2k + \frac{3t}{1-\frac{t}{k}}$ and $1 \leqslant t < k$, then $\delta_t(\mathcal{F}) \leqslant \binom{n-t-1}{k-t-1}$.

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Degree versions for non-trivial families

Theorem (Frankl, Han, Huang, Zhao, 2017+)

Let $k \geqslant 4$ and $n \geqslant ck^2$, where $c = 30$ for $k = 4, 5$, and $c = 4$ for $k \geqslant 6$. Then the minimum degree of any non-trivial intersecting family is at most $\binom{n-2}{k-2} - \binom{n-k-2}{k-2}$.

Question: extend this result for n linear in k, or even for $n \geqslant 2k + 1$.

Theorem (Kupavskii, 2017+)

If $t = 1$, $n \geqslant 2k + 5$, and $k \geqslant 35$, or $1 < t \leqslant \frac{k}{4} - 2$, $n \geqslant 2k + 14t$, then for any non-trivial intersecting family $\mathcal{F}$ of k-subsets of $[n]$ we have $\delta_t(\mathcal{F}) \leqslant \binom{n-t-1}{k-t-1} - \binom{n-k-t-1}{k-t-1}$.

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Further questions

Han-Kohayakawa, Mubayi-Kostochka: study of non-trivial families that are not subfamilies of the Hilton-Milner families.

Their results may be strengthened and generalized using our methods.

What one can say about the structure of the families with diversity bigger than $\binom{n-3}{k-2}$?

Conjecture (Frankl)

If $n \geqslant 3k$ and $\mathcal{F}$ is an intersecting family of k-subsets of $[n]$, then $\gamma(\mathcal{F}) \leqslant \binom{n-3}{k-2}$.

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