Families with forbidden subconfigurations

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Partitions

A subset $\mathcal{F} \subset 2^{[n]}$ is called a *family*.

A partition in $\mathcal{F}$: two disjoint sets $F_1, F_2 \in \mathcal{F}$, such that

$$F_1 \cup F_2 \in \mathcal{F}.$$

$$p(n) := \max\{|\mathcal{F}| : \mathcal{F} \subset 2^{[n]}, \mathcal{F} \text{ is partition-free}\}.$$

Partitions

For $n = 3m + i, \ \ i = 0, 1, 2,$ the family

$$\mathcal{K}(n) := \{K \subset [n] : m + 1 \leqslant |K| \leqslant 2m + 1\}$$

does not contain a partition.

Theorem (Kleitman, 1968)

$$p(3m + 1) = |\mathcal{K}(3m + 1)| = \sum_{m+1 \leqslant t \leqslant 2m+1} \binom{n}{t}.$$

Kleitman conjectured that $\boxed{p(n) = |\mathcal{K}(n)|}$ for any n.

Theorem 1 (P. Frankl, AK, 2017)

The conjecture is true. Moreover, we know all extremal families.

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Matchings

The matching number $\nu(\mathcal{F})$: the maximum number of pairwise disjoint members of $\mathcal{F}$.

$$e(n,s) := \max\left\{|\mathcal{F}| : \mathcal{F} \subset 2^{[n]}, \quad \nu(\mathcal{F}) < s\right\}.$$

$$e_k(n,s) := \max\left\{|\mathcal{F}| : \mathcal{F} \subset \binom{[n]}{k}, \ \nu(\mathcal{F}) < s\right\}.$$

Matchings

The case $s = 2$ corresponds to intersecting families.

Theorem (Erdős-Ko-Rado, 1938-1961)

$$e(n,2) = 2^{n-1},$$

$$e_k(n,2) = \binom{n-1}{k-1} \quad \text{for} \quad n \geqslant 2k.$$

Matchings. The non-uniform case

For $n < sm$ the family

$$\mathcal{B}(n,m) := \binom{[n]}{\geqslant m} := \{H \subset [n] : |H| \geqslant m\}$$

does not contain s pairwise disjoint sets.

Conjecture (Erdős, 1960's)

For $n = sm - 1$ we have $\quad e(n,s) = |\mathcal{B}(n,m)|$.

Theorem (Kleitman, 1966)

$$e(sm-1,s) = |\mathcal{B}(n,m)|,$$

$$e(sm,s) = \binom{sm-1}{m} + \sum_{m+1 \leqslant t \leqslant sm} \binom{sm}{t} \qquad (= 2e(sm-1,s)).$$

Matchings. The non-uniform case

Problem (Kleitman, 1966)

Determine $e(n, s)$ for other values of n.

Very little progress over 50 years...

Theorem (Quinn, 1987)

$$e(3m+1, 3) = \binom{3m}{m-1} + \sum_{m+1 \leqslant t \leqslant 3m+1} \binom{3m+1}{t}.$$

Unfortunately, it was not published in a refereed journal.

Construction

Let $n = sm + s - l$, $0 < l \leqslant s$.

$$\mathcal{P}(s,m,l) := \left\{P \subset 2^{[n]} : |P| + |P \cap [l-1]| \geqslant m+1\right\}.$$

$\nu(\mathcal{P}(s,m,l)) < s$: for disjoint $F_1, \dots, F_s$ we have

$$sm + s - 1 \geqslant \sum_{i=1}^{s} |F_i| + |F_i \cap [l-1]| \geqslant sm + s.$$

Theorem 2 (P. Frankl, AK, 2016)

$e(sm + s - l, s) = |\mathcal{P}(s,m,l)|$ holds for

(2) $l = 2$,

(1) $s \geqslant lm + 3l + 3$.

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Remark. Cross-dependent families

Families $\mathcal{F}_1, \ldots, \mathcal{F}_s$ are *cross-dependent*,

if there are no pairwise disjoint $F_i \in \mathcal{F}_i$, $i \in [s]$.

Theorem 3 (P. Frankl, AK, 2016)

Let $n = sm + s - l$ with $1 \leqslant l \leqslant s$. Then

$$\sum_{i=1}^{s} |\mathcal{F}_i| \leqslant (l-1)\binom{n}{m} + s \sum_{t \geqslant m+1} \binom{n}{t}.$$

This is tight: take $\mathcal{F}_1 = \ldots = \mathcal{F}_{l-1} = \binom{[n]}{\geqslant m}$

and $\mathcal{F}_l = \ldots = \mathcal{F}_s = \binom{[n]}{\geqslant m+1}$.

Matchings. Proof for $n = 3m + 2$

Theorem

Put $n = 3m + 2$. Any family $\mathcal{F} \subset 2^{[n]}$ with no three pairwise disjoint sets satisfies

$$|\mathcal{F}| \leqslant \sum_{t=m+1}^{n} \binom{n}{m}.$$

Step 1. An auxiliary family $\mathcal{H}$. It consists of:

(a) Three pairwise disjoint m-sets $H^1, H^2, H^3 \subset [3m + 2]$.

(Put $\{x, y\} := [3m + 2] \setminus \cup H^i$.)

(b) Three maximal chains $\{\emptyset\} =: H^i_0 \subset H^i_1 \subset \ldots \subset H^i_m := H^i$.

(c) All sets of the type $H^i \cup A$, where $A \subset \{x, y\}$.

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Step 2. Weights and averaging. The weights are as follows:

(a) Each set $H \in \mathcal{H}$ of size $t \neq m+1$ gets weight $w(H) = \binom{n}{t}$.

(b) Each set $U \in \mathcal{H}$ of size $m+1$ gets weight $w(U) = \frac{1}{2}\binom{n}{m+1}$.

The sum of weights of all t-sets in $\mathcal{H}$ is equal to $3\binom{n}{t}$.

Need to prove that $\sum_{H \in \mathcal{H} \setminus \mathcal{F}} w(H) \geqslant 3 \sum_{t=0}^{m} \binom{n}{t}$ for any choice of $\mathcal{H}$.

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$$\sum_{H \in \mathcal{H} \setminus \mathcal{F}} w(H) \geqslant 3 \sum_{t=0}^{m} \binom{n}{t}. \tag{1}$$

Why (1) implies the theorem? Choose $\mathcal{H}$ "at random".

For any $H \in \mathcal{H}$ of size t $\quad \Pr[H \notin \mathcal{F}] = \frac{|\binom{[n]}{t} \setminus \mathcal{F}|}{\binom{n}{t}}$.

For $t \leqslant m + 2$ we have $\quad \mathrm{E}\left[\sum_{H \in \mathcal{H} \cap \binom{[n]}{t} \setminus \mathcal{F}} w(H)\right] = 3|\binom{[n]}{t} \setminus \mathcal{F}|$.

Therefore, by (1) we get

$$3\sum_{t=0}^{m} \binom{n}{t} \leqslant \mathrm{E}\Big[\sum_{H \in \mathcal{H} \setminus \mathcal{F}} w(H)\Big] = 3 \sum_{t=0}^{m+2} \Big|\binom{[n]}{t} \setminus \mathcal{F}\Big| \leqslant 3|2^{[n]} \setminus \mathcal{F}|.$$

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Proof of the $n = 3m + 2$ case

Need to prove $$\sum_{H \in \mathcal{H} \setminus \mathcal{F}} w(H) \geqslant 3 \sum_{t=0}^{m} \binom{n}{t}.$$

Step 3. Analysis of $\mathcal{F} \cap \mathcal{H}$.

We can assume that $\mathcal{F}$ is closed upwards.

Case 1: For each $i \in [3]$ $H^i \notin \mathcal{F}$. Then all H_t^i are missing, $t = 0, \ldots, m$. We are done.

Case 2: $H^1, H^2 \in \mathcal{F}, H^3 \notin \mathcal{F}$. Then $H^3 \cup \{x, y\} \notin \mathcal{F}$. We have

$$\sum_{H \in \mathcal{H} \setminus \mathcal{F}} w(H) \geqslant \sum_{t=0}^{m+2} \binom{n}{t} > 3 \sum_{t=0}^{m} \binom{n}{t}.$$

We use that $\binom{n}{m+2} \geqslant \binom{n}{m+1} > \sum_{t=0}^{m} \binom{n}{t}$.

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Proof of the $n = 3m + 2$ case

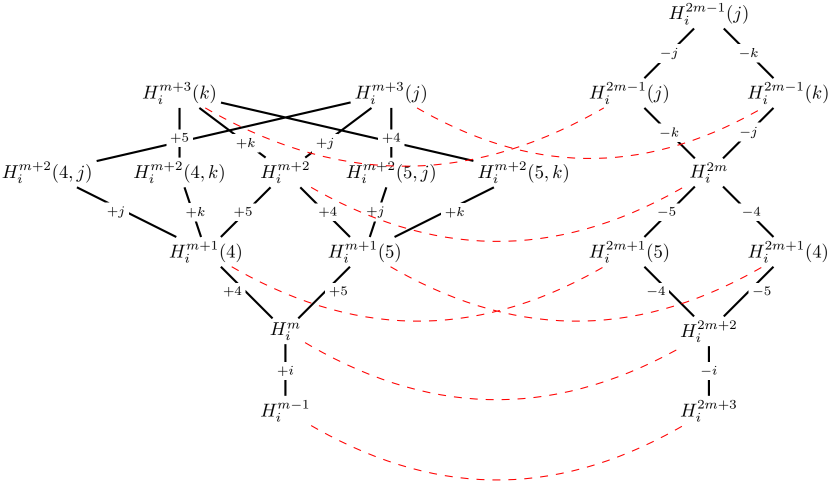
$$\sum_{H \in \mathcal{H} \setminus \mathcal{F}} w(H) \geqslant 3 \sum_{t=0}^{m} \binom{n}{t}.$$

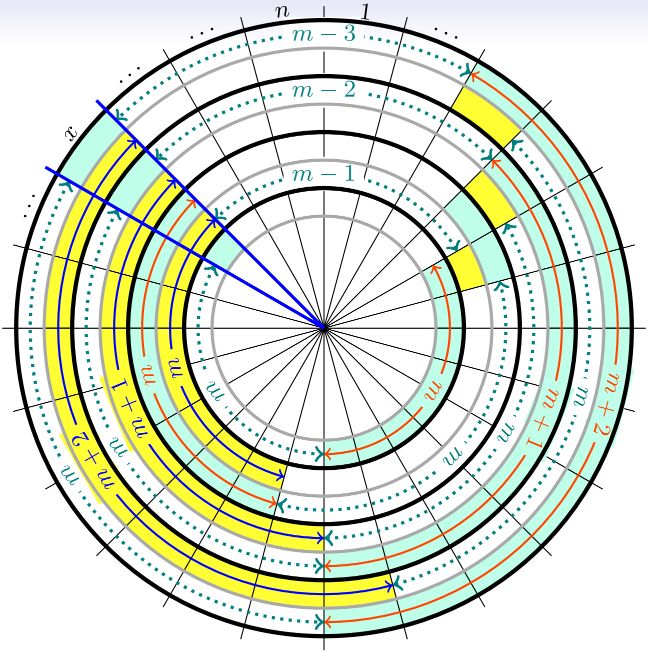
Case 3: $H^1 \in \mathcal{F}, H^2, H^3 \notin \mathcal{F}$. Then in each pair

$$\big(H^2 \cup \{x\}, H^3 \cup \{y\}\big), \qquad \big(H^2 \cup \{y\}, H^3 \cup \{x\}\big)$$

one set is missing. We get

$$\sum_{H \in \mathcal{H} \setminus \mathcal{F}} w(H) \geqslant \binom{n}{m+1} + 2 \sum_{t=0}^{m} \binom{n}{t} > 3 \sum_{t=0}^{m} \binom{n}{t}.$$


$H_i^{2m-1}(j)$
$-j$
$-k$
$H_i^{m+3}(k)$
$H_i^{m+3}(j)$
$H_i^{2m-1}(j)$
$H_i^{2m-1}(k)$
$+5$
$+k$
$+j$
$+4$
$-k$
$-j$
$H_i^{m+2}(4,j)$
$H_i^{m+2}(4,k)$
H_i^{m+2}
$H_i^{m+2}(5,j)$
$H_i^{m+2}(5,k)$
H_i^{2m}
$+j$
$+k$
$+5$
$+4$
$+j$
$+k$
-5
-4
$H_i^{m+1}(4)$
$H_i^{m+1}(5)$
$H_i^{2m+1}(5)$
$H_i^{2m+1}(4)$
$+4$
$+5$
-4
-5
H_i^m
H_i^{2m+2}
$+i$
$-i$
H_i^{m-1}
H_i^{2m+3}



n
1
x
m − 3
m − 2
m − 1
m
m + 1
m + 2

Matchings. The uniform case

How to construct a large family $\mathcal{A} \subset \binom{[n]}{k}$, satisfying $\nu(\mathcal{A}) < s$?

$$\mathcal{A}_1^{(k)}(n,s) := \left\{A \in \binom{[n]}{k} : A \cap [s-1] \neq \emptyset\right\}, \quad \mathcal{A}_k^{(k)}(n,s) := \binom{[sk-1]}{k}.$$

Erdős Matching Conjecture, 1965

For $n \geqslant sk$ we have

$$e_k(n,s) = \max\{|\mathcal{A}_1^{(k)}(n,s)|, |\mathcal{A}_k^{(k)}(n,s)|\}.$$

True for $k \leqslant 3$ (Erdős and Gallai; Łuczak and Mieczkowska; Frankl).

$$e_k(n,s+1) = \binom{n}{k} - \binom{n-s+1}{k} \qquad \text{for} \quad n \geqslant (2s-1)k - s \quad \text{(Frankl)}.$$

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Stability Results

The covering number $\tau(\mathcal{H})$ of a family is the minimum of $|T|$ over all T satisfying $T \cap H \neq \emptyset$ for all $H \in \mathcal{H}$.

Hilton-Milner, 1967

Let $n \geqslant 2k$ and $\mathcal{F} \subset \binom{[n]}{k}$ satisfy $\nu(\mathcal{F}) < 2$ and $\tau(\mathcal{F}) \geqslant 2$. Then

$$|\mathcal{F}| \leqslant \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1 \qquad \text{holds.}$$

Stability Results

Theorem 4 (P. Frankl, AK, 2016)
Assume that $\nu(\mathcal{F}) < s,\ \tau(\mathcal{F}) \geqslant s$. Then the following holds:

$$|\mathcal{F}| \leqslant \binom{n}{k} - \binom{n-s+1}{k} - \binom{n-s+1-k}{k-1} + 1,$$

provided $k \geqslant 3,\ \ n \geqslant \big(2 + o(1)\big) sk$, where $o(1)$ depends on s only.

Known to be true for $n > 2k^3 s$: Bollobás, Daykin and Erdős (1976).

Implications for other problems: anti-Ramsey type questions, non-uniform families with no large matchings, degree versions.

Open problems.

Let $\alpha_1 \geqslant \ldots \geqslant \alpha_n \geqslant 0$ be reals, $\sum_i \alpha_i < s$. Put $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$.

$$\mathcal{F}(\boldsymbol{\alpha}) := \{F \in 2^{[n]} : \sum_{i \in F} \alpha_i \geqslant 1\}.$$

Then $\nu(\mathcal{F}(\boldsymbol{\alpha}) < s$ holds. Also $\mathcal{F}(\boldsymbol{\alpha}) = \{0,1\}^n \cap \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x}, \boldsymbol{\alpha} \rangle \geqslant 1\}$.

Conjecture (P. Frankl, AK, 2016)

For any n, s the maximum of $e(n,s)$ (or $e_k(n,s)$) is attained on the family $\mathcal{F}(\boldsymbol{\alpha})$ for suitable $\boldsymbol{\alpha} \in \mathbb{R}^n$.

Non-uniform case:

Maximum size of families without certain structures involving disjoint sets (e.g., r-partition free families, introduced by Frankl in 1977).