Hypergraphs with no s pairwise disjoint edges

Andrey Kupavskii

DCG, EPFL

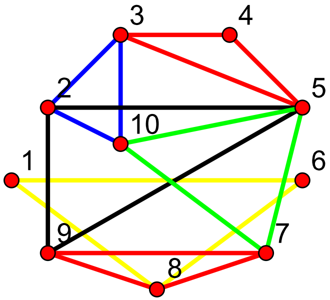
Joint work with Peter Frankl

Journées Graphes et Algorithmes 2016

Definitions

A subset $\mathcal{F} \subset 2^{[n]}$ is called a *family*.

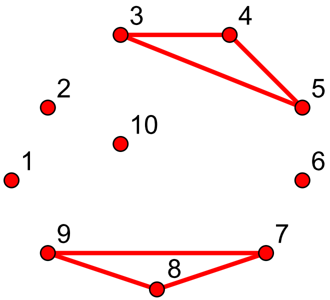
$\nu(\mathcal{F})$: the maximum number of pairwise disjoint members of $\mathcal{F}$.



Definitions

A subset $\mathcal{F} \subset 2^{[n]}$ is called a *family*.

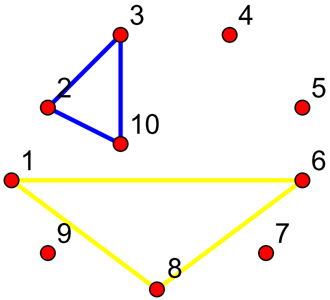
$\nu(\mathcal{F})$: the maximum number of pairwise disjoint members of $\mathcal{F}$.



Definitions

A subset $\mathcal{F} \subset 2^{[n]}$ is called a *family*.

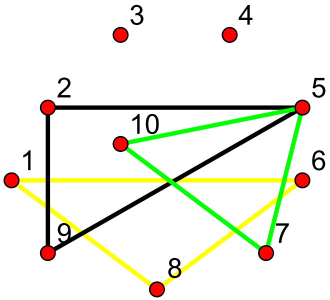
$\nu(\mathcal{F})$: the maximum number of pairwise disjoint members of $\mathcal{F}$.



Definitions

A subset $\mathcal{F} \subset 2^{[n]}$ is called a *family*.

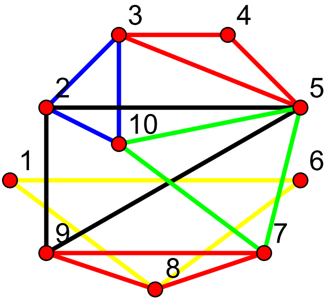
$\nu(\mathcal{F})$: the maximum number of pairwise disjoint members of $\mathcal{F}$.



Definitions

A subset $\mathcal{F} \subset 2^{[n]}$ is called a *family*.

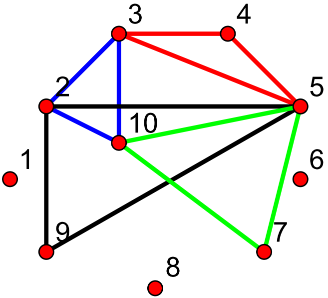
$\nu(\mathcal{F})$: the maximum number of pairwise disjoint members of $\mathcal{F}$.



Definitions

A subset $\mathcal{F} \subset 2^{[n]}$ is called a *family*.

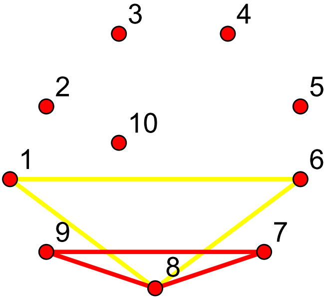
$\nu(\mathcal{F})$: the maximum number of pairwise disjoint members of $\mathcal{F}$.



Definitions

A subset $\mathcal{F} \subset 2^{[n]}$ is called a *family*.

$\nu(\mathcal{F})$: the maximum number of pairwise disjoint members of $\mathcal{F}$.



Definitions

A subset $\mathcal{F} \subset 2^{[n]}$ is called a *family*.

$\nu(\mathcal{F})$: the maximum number of pairwise disjoint members of $\mathcal{F}$.

$$e(n, s) := \max\Big\{|\mathcal{F}| : \mathcal{F} \subset 2^{[n]}, \nu(\mathcal{F}) < s\Big\}.$$

$$e_k(n, s) := \max\left\{|\mathcal{F}| : \mathcal{F} \subset \binom{[n]}{k}, \nu(\mathcal{F}) < s\right\}.$$

The case $s = 2$ corresponds to intersecting families.

Theorem (Erdős-Ko-Rado, 1961)

$$e(n,2) = 2^{n-1}.$$

$$e_k(n,2) = \binom{n-1}{k-1} \quad \text{for} \quad n \geqslant 2k.$$

The non-uniform case

For $n < sm$ the family

$$\mathcal{B}(n,m) := \binom{[n]}{\geqslant m} := \{H \subset [n] : |H| \geqslant m\}$$

does not contain s pairwise disjoint sets.

Conjecture (Erdős, 1960's)

For $n = sm - 1$ $\mathcal{B}(n,m)$ is the largest family with matching number $< s$.

Theorem (Kleitman, 1966)

$$e(sm-1,s) = |\mathcal{B}(n,m)| = \sum_{m \leqslant t \leqslant sm-1} \binom{sm-1}{t},$$

$$e(sm,s) = \binom{sm-1}{m} + \sum_{m+1 \leqslant t \leqslant sm} \binom{sm}{t} \quad (= 2e(sm-1,s)).$$

The non-uniform case

Problem (Kleitman, 1966)

Determine $e(n, s)$ for other values of n.

Very little progress over 50 years...

Theorem (Quinn, 1986)

$$e(3m+1, 3) = \binom{3m}{m-1} + \sum_{m+1 \leqslant t \leqslant 3m+1} \binom{3m+1}{t}.$$

Construction

Let $n = sm + s - l$, $0 < l \leqslant s$.

$$\mathcal{P}(s,m,l) := \left\{ P \subset 2^{[n]} : |P| + |P \cap [l-1]| \geqslant m+1 \right\}.$$

$\nu(\mathcal{P}(s,m,l)) < s$: for disjoint $F_1, \ldots, F_s$ we have

$$sm + s - 1 \geqslant \sum_{i=1}^{s} |F_i| + |F_i \cap [l-1]| \geqslant sm + s.$$

Theorem (P. Frankl, AK, 2016)

$e(sm + s - l, s) = |\mathcal{P}(s,m,l)|$ holds for

$$(i)\ l = 2,$$
$$(ii)\ s \geqslant lm + 3l + 3.$$

Construction

Let $n = sm + s - l$, $0 < l \leqslant s$.

$$\mathcal{P}(s,m,l) := \left\{P \subset 2^{[n]} : |P| + |P \cap [l-1]| \geqslant m+1\right\}.$$

$\nu(\mathcal{P}(s,m,l)) < s$: for disjoint $F_1, \ldots, F_s$ we have

$$sm + s - 1 \geqslant \sum_{i=1}^{s} |F_i| + |F_i \cap [l-1]| \geqslant sm + s.$$

Theorem (P. Frankl, AK, 2016)
$e(sm+s-l, s) = |\mathcal{P}(s,m,l)|$ holds for

$$(i)\ l = 2,$$
$$(ii)\ s \geqslant lm + 3l + 3.$$

Statement for cross-dependent families

Families $\mathcal{F}_1, \dots, \mathcal{F}_s$ are *cross-dependent*, if there are no pairwise disjoint $F_i \in \mathcal{F}_i$, $i = 1, \dots, s$.

Theorem (P. Frankl, AK, 2016)

Let $n = sm + s - l$ with $1 \leqslant l \leqslant s$. Then

$$\sum_{i=1}^{s} |\mathcal{F}_i| \leqslant (l-1)\binom{n}{m} + s \sum_{t \geqslant m+1} \binom{n}{t}.$$

Uniform case

How to construct a large family $\mathcal{A} \subset \binom{[n]}{k}$, satisfying $\nu(\mathcal{A}) < s$?

$$\mathcal{A}_1^{(k)}(n,s) := \left\{ A \in \binom{[n]}{k} : A \cap [s-1] \neq \emptyset \right\}, \quad \mathcal{A}_k^{(k)}(n,s) := \binom{[sk-1]}{k}.$$

Erdős Matching Conjecture, 1965

For $n \geqslant sk$ we have

$$e_k(n,s) = \max\{|\mathcal{A}_1^{(k)}(n,s)|, |\mathcal{A}_k^{(k)}(n,s)|\}.$$

True for $k \leqslant 3$ (Erdős and Gallai; Łuczak and Mieczkowska; Frankl).

$$e_k(n,s+1) = \binom{n}{k} - \binom{n-s+1}{k} \qquad \text{for} \quad n \geqslant (2s-1)k - s \quad \text{(Frankl).} \tag{1}$$

Uniform case

How to construct a large family $\mathcal{A} \subset \binom{[n]}{k}$, satisfying $\nu(\mathcal{A}) < s$?

$$\mathcal{A}_1^{(k)}(n,s) := \left\{A \in \binom{[n]}{k} : A \cap [s-1] \neq \emptyset\right\}, \quad \mathcal{A}_k^{(k)}(n,s) := \binom{[sk-1]}{k}.$$

Erdős Matching Conjecture, 1965

For $n \geqslant sk$ we have

$$e_k(n,s) = \max\{|\mathcal{A}_1^{(k)}(n,s)|, |\mathcal{A}_k^{(k)}(n,s)|\}.$$

True for $k \leqslant 3$ (Erdős and Gallai; Łuczak and Mieczkowska; Frankl).

$$e_k(n,s{+}1) = \binom{n}{k} - \binom{n-s+1}{k} \qquad \text{for} \qquad n \geqslant (2s-1)k - s \quad \text{(Frankl)}. \tag{1}$$

Uniform case

How to construct a large family $\mathcal{A} \subset \binom{[n]}{k}$, satisfying $\nu(\mathcal{A}) < s$?

$$\mathcal{A}_1^{(k)}(n,s) := \left\{A \in \binom{[n]}{k} : A \cap [s-1] \neq \emptyset\right\}, \quad \mathcal{A}_k^{(k)}(n,s) := \binom{[sk-1]}{k}.$$

Erdős Matching Conjecture, 1965

For $n \geqslant sk$ we have

$$e_k(n,s) = \max\{|\mathcal{A}_1^{(k)}(n,s)|, |\mathcal{A}_k^{(k)}(n,s)|\}.$$

True for $k \leqslant 3$ (Erdős and Gallai; Łuczak and Mieczkowska; Frankl).

$$e_k(n,s{+}1) = \binom{n}{k} - \binom{n-s+1}{k} \qquad \text{for} \qquad n \geqslant (2s-1)k - s \quad \text{(Frankl)}. \tag{1}$$

Stability Results

The covering number $\tau(\mathcal{H})$ of a hypergraph is the minimum of $|T|$ over all T satisfying $T \cap H \neq \emptyset$ for all $H \in \mathcal{H}$.

Hilton-Milner, 1967

Let $n \geqslant 2k$ and $\mathcal{F} \subset \binom{[n]}{k}$ satisfy $\nu(\mathcal{F}) < 2$ and $\tau(\mathcal{F}) \geqslant 2$. Then

$$|\mathcal{F}| \leqslant \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1 \qquad \text{holds.}$$

Theorem (P. Frankl, AK, 2016)

Assume that $\nu(\mathcal{F}) < s, \tau(\mathcal{F}) \geqslant s$. Then the following holds:

$$|\mathcal{F}| \leqslant \binom{n}{k} - \binom{n-s+1}{k} - \binom{n-s+1-k}{k-1} + 1,$$

provided $k \geqslant 3, n \geqslant (2+o(1))sk$, where $o(1)$ depends on s only.

Known to be true for $n > 2k^3 s$: Bollobás, Daykin and Erdős (1976).

Stability Results

The covering number $\tau(\mathcal{H})$ of a hypergraph is the minimum of $|T|$ over all T satisfying $T \cap H \neq \emptyset$ for all $H \in \mathcal{H}$.

Hilton-Milner, 1967

Let $n \geqslant 2k$ and $\mathcal{F} \subset \binom{[n]}{k}$ satisfy $\nu(\mathcal{F}) < 2$ and $\tau(\mathcal{F}) \geqslant 2$. Then

$$|\mathcal{F}| \leqslant \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1 \qquad \text{holds.}$$

Theorem (P. Frankl, AK, 2016)

Assume that $\nu(\mathcal{F}) < s, \tau(\mathcal{F}) \geqslant s$. Then the following holds:

$$|\mathcal{F}| \leqslant \binom{n}{k} - \binom{n-s+1}{k} - \binom{n-s+1-k}{k-1} + 1,$$

provided $k \geqslant 3, n \geqslant \big(2 + o(1)\big)sk$, where $o(1)$ depends on s only.

Known to be true for $n > 2k^3 s$: Bollobás, Daykin and Erdős (1976).

Open problems.

Let $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_n \geqslant 0$ be reals, $\sum_i \alpha_i < s$. Put $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$.

$$\mathcal{F}(\boldsymbol{\alpha}) := \{F \in 2^{[n]} : \sum_{i \in F} \alpha_i \geqslant 1\}.$$

Then $\nu(\mathcal{F}(\boldsymbol{\alpha}) < s$ holds. Also $\mathcal{F}(\boldsymbol{\alpha}) = \{0,1\}^n \cap \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x}, \boldsymbol{\alpha} \rangle \geqslant 1\}$.

Conjecture (P. Frankl, AK)

For any n, s the maximum of $e(n, s)$ (or $e_k(n, s)$) is attained on the family $\mathcal{F}(\boldsymbol{\alpha})$ for suitable $\boldsymbol{\alpha} \in \mathbb{R}^n$.