Colorings of uniform hypergraphs with large girth and applications*

Andrey Kupavskii[†], Dmitry Shabanov[‡]

Abstract

The work deals with a combinatorial problem concerning colorings of uniform hypergraphs with large girth. We prove that if H is an n-uniform non-r-colorable simple hypergraph then its maximum edge degree $\Delta(H)$ satisfies the inequality

$$\Delta(H) \geqslant c \cdot r^{n-1} \frac{n(\ln\ln n)^2}{\ln n}$$

for some absolute constant $c > 0$.

As an application of our probabilistic technique we establish a lower bound for the classical Van der Waerden number $W(n,r)$, the minimum natural N such that in arbitrary coloring of the set of integers $\{1,\ldots,N\}$ with r colors there exists a monochromatic arithmetic progression of length n. We prove that

$$W(n,r) \geqslant c \cdot r^{n-1} \frac{(\ln\ln n)^2}{\ln n}.$$

Keywords: colorings of hypergraphs, hypergraphs with large girth, simple hypergraphs, random recoloring method, Van der Waerden number.

MSC codes: 05C15, 05D40, 05D10.

1 Introduction

The work deals with an extremal combinatorial problem concerning colorings of uniform hypergraphs with large girth. Recall the main definitions.

One of the basic facts of graph coloring theory is that the chromatic number of an arbitrary graph with maximum vertex degree d is at most $d+1$. A natural generalization of this fact for

*This work was supported by Russian Foundation of Fundamental Research (grant № 12-01-00683-a), by the program "Leading Scientific Schools" (grant no. NSh-2519.2012.1).

[†]Moscow Institute of Physics and Technology, Department of Discrete Mathematics; Ecole Polytechnique Fédérale de Lausanne. E-mail: kupavskii@ya.ru

[‡]Lomonosov Moscow State University, Faculty of Mechanics and Mathematics, Department of Probability Theory, 119991, Leninskie gory 1, Moscow, Russia. E-mail: dm.shabanov.msu@gmail.com

edge degrees can be stated as follows: if $\Delta(G) \leqslant 2r-3$ then G is r-colorable. Both bounds are tight since we have an equality for complete graphs.

The situation in the case of uniform hypergraphs is much more complicated. There are unknown sharp quantitative bounds for the chromatic number in terms of the maximum edge (or vertex) degree. A fundamental result in this field was obtained by P. Erdős and L. Lovász in their classical paper [5]. They showed that if H is an arbitrary n-uniform hypergraph with maximum edge degree $\Delta(H)$ satisfying

$$\Delta(H) \leqslant \frac{1}{e} r^{n-1}, \tag{1}$$

then H is r-colorable. This result does not provide a tight bound for the maximum edge degree. The restriction (1) was successively weakened for different values of the parameter r in a series of papers. The results were obtained by Radhakrishnan and Srinivasan (see [15]) for $r=2$, Shabanov (see [18]) for $r=3$, Kostochka, Kumbhat and Rödl (see [11]) for $r>3$. Recently for fixed $r>2$, the last two bounds were improved by Cherkashin and Kozik (see [4]).

First analogue of the Erdős–Lovász statement for simple hypergraphs was proved by Szabó in 1990 (see [22]). Actually he gave a lower bound for the maximum vertex degree in an arbitrary n-uniform non-2-colorable simple hypergraph. Later by using a similar technique Kostochka and Kumbhat established the following extension of his result.

Theorem 1. (A.V. Kostochka, M. Kumbhat, [10]) *For any $\varepsilon>0$ and $r \geqslant 2$ there exists $n_0 = n_0(\varepsilon, r)$ such that for any $n > n_0$, every n-uniform simple hypergraph H with maximum edge degree at most*

$$\Delta(H) \leqslant r^{n-1} n^{1-\varepsilon}, \tag{2}$$

is r-colorable.

Since the parameter ε can be taken arbitrarily small in Theorem 1 one can replace it in (2) by a function $\varepsilon(n,r)$. In their final comment in [10] Kostochka and Kumbhat asserted that for fixed r, it is possible to take $\varepsilon(n,r) = \Theta\left(\frac{\ln\ln\ln n}{\ln\ln n}\right)$. However, the calculations in their proof gave a stronger lower bound $\varepsilon(n,r) = \Omega\left(\sqrt[4]{\ln r/\ln n}\right)$. Theorem 1 was refined by Shabanov in [19], where he proved that, for fixed r, the function $\varepsilon(n,r)$ can be taken of an order $\sqrt{\ln\ln n/\ln n}$. Note that both results do not improve even the classical bound of Erdős and Lovász for all sufficiently large r.

The main result of the current paper is the following Erdős–Lovász–type theorem for simple hypergraphs.

Theorem 2. *Suppose $n \geqslant 9$, $r \geqslant 2$ and H is an n-uniform simple hypergraph. There exists an absolute constant $c>0$ such that if*

$$\Delta(H) \leqslant c \cdot r^{n-1} \frac{n(\ln\ln n)^2}{\ln n} \tag{3}$$

then H is r-colorable.

The obtained bound (3) improves the above results by Kostochka–Kumbhat and Shabanov. Indeed, we have the bound of the same type: $r^{n-1}n^{1-\varepsilon}$ with better $\varepsilon = \frac{\ln\ln n}{\ln n}(1+o(1))$. It is $n^{\frac{(\ln n)(\ln r)}{(\ln\ln n)^2}}$ times smaller than the best known upper bound given by Kostochka and Rödl, who proved (see [12]) that, for any $n, r \geqslant 2$, there exists an n-uniform non-r-colorable simple hypergraph with

$$\Delta(H) \leqslant n^2 r^{n-1} \ln r.$$

The extremal results concerning colorings of uniform hypergraphs can be often applied in various problems of Ramsey theory. In the next section we shall discuss the application of our main result to estimating the Van der Waerden number.

2 Bounds for the Van der Waerden number

In 1927 B. Van der Waerden proved (see [23]) his famous theorem on arithmetic progressions. It states that for any integers $n \geqslant 3$ and $r \geqslant 2$, there exists the minimum number $W(n,r)$ such that any coloring with r colors of the set of integers $\{1, \ldots, W(n,r)\}$ contains a monochromatic arithmetic progression of length n. The function $W(n,r)$ from the Van der Waerden theorem is called the *Van der Waerden number* or the *Van der Waerden function*.

The best known upper bounds for $W(n,r)$ are derived from the results concerning densities of sets of integers without long arithmetic progressions. Let for any $N > n$, $r_n(N)$ denote the maximum density of a subset of $\{1, \ldots, N\}$ without long arithmetic progressions, i.e.

$$r_n(N) = \frac{1}{N} \max\left\{|A| : A \subset \{1, \ldots, N\},\ A \text{ does not contain APs of length } n\right\}.$$

It is easy to understand the following relation between the Van der Waerden number and $r_n(N)$:

$$\text{if } r_n(N) < \frac{1}{r},\ \text{then } W(n,r) \leqslant N.$$

Indeed, every r-coloring of $\{1, \ldots, N\}$ has a color class of a size at least N/r. If $r_n(N) < \frac{1}{r}$ then its density is greater than $r_n(N)$ and by the definition of $r_n(N)$ this color class should contain an arithmetic progression of length n. So, the upper bounds for $r_n(N)$ imply upper bounds for $W(n,r)$.

For $n = 3$, the best record for $r_n(N)$ was obtained by Sanders in [16], who showed that

$$r_3(N) = O\left(\frac{(\ln\ln N)^5}{\ln N}\right).$$

This estimate immediately implies the following upper bound for $W(3,r)$:

$$W(3,r) \leqslant \exp\left\{c\, r(\ln r)^5\right\}$$

where $c > 0$ is some absolute constant.

For the case $n = 4$, Green and Tao proved (see [9]) that

$$r_4(N) \leqslant \exp\left\{-c\sqrt{\ln\ln N}\right\}.$$

As a corollary we get the best known upper bound for $W(4,r)$ which is a double exponent of $(\ln r)^2$:

$$W(4,r) \leqslant e^{e^{c(\ln r)^2}}.$$

Finally, in general case $(n>4)$ the best upper bound for $r_n(N)$ was given by Gowers in his famous paper [7]:

$$r_n(N) < \frac{1}{(\log_2 \log_2 N)^{c_n}}, \text{ where } c_n = 2^{-2^{n+9}}.$$

So, the best general upper estimate for the Van der Waerden number is the following tower of six numbers:

$$W(n,r) \leqslant 2^{2^{r^{2^{2^{n+9}}}}}. \tag{4}$$

We finished with the upper bounds and now proceed to the history of estimating $W(n,r)$ from below.

First nontrivial lower bound for the Van der Waerden number $W(n,r)$ was obtained by Erdős and Rado in 1952 (see [6]). By using simple probabilistic approach they established that

$$W(n,r) \geqslant \sqrt{2(n-1)r^{n-1}}. \tag{5}$$

In 1960 this bound was improved for large values of r (in comparison with n) by Moser (see [13]), who gave an explicit construction of r-coloring of the set integers without long arithmetic progressions and derived that

$$W(n,r) \geqslant n \cdot r^{c \ln r} \tag{6}$$

for some absolute constant $c>0$. The result (5) of Erdős and Rado was asymptotically improved by Schmidt (see [17]) in 1962. He showed that there exists an absolute constant $c>0$ such that

$$W(n,r) \geqslant r^{n-c\sqrt{n\ln n}}. \tag{7}$$

In the particular case when p is a prime number and $r=2$ E. Berlekamp established (see [3]) the relation

$$W(p+1,2) > p2^p. \tag{8}$$

Further advances concerning lower bounds for $W(n,r)$ were made by the help of the results and methods of hypergraph coloring theory. How is the Van der Waerden number connected with colorings of hypergraphs? For any integers $N>n$, consider a hypergraph $H_n(N) = ([N], E_n(N))$, where $[N]=\{1,2,\ldots,N\}$ and $E_n(N)$ is a collection of all arithmetic progressions of length n contained in $[N]$. Clearly $H_n(N)$ is an n-uniform hypergraph. Note that $H_N(n)$ is an induced subhypergraph in $H_n(N+1)$. Therefore the chromatic number of $H_n(N+1)$ is least as the chromatic number of $H_n(N)$:

$$\chi\left(H_n(N)\right) \leqslant \chi\left(H_n(N+1)\right).$$

And it is easy to see that in terms of hypergraph coloring theory an equivalent definition of the Van der Waerden number can be formulated as follows:

$$W(n,r) = \min\left\{N: \ \chi\left(H_n(N)\right) > r\right\}.$$

Thus, for establishing the inequality $W(n,r) > N$ for some N we have to show that the hypergraph of arithmetic progressions $H_n(N)$ is r-colorable. One of the natural ways for this purpose is to use quantitative relations between the chromatic number number and other hypergraph characteristics. For example, applying statement 1 of Erdős and Lovász one can easily get the following bound

$$W(n,r) \geqslant \frac{r^{n-1}}{en}\left(1-\frac{1}{n}\right). \tag{9}$$

Indeed, for any $x \in [N]$, there is at most $n(N-1)/(n-1)$ arithmetic progressions of length n from $[N]$ containing x since any such progression is uniquely defined by the position of x in the progression and by the difference of the progression. Thus, if

$$N \leqslant \frac{r^{n-1}}{4n}\left(1-\frac{1}{n}\right)+1,$$

then the maximum edge degree of $H_n(N)$ does not exceed r^{n-1}/e and due to Theorem **??** $H_n(N)$ is r-colorable. Therefore the bound (9) is proved. Note that it improves the previous result by Schmidt (7).

In connection with estimating the Van der Waerden number the results of the Erdős–Lovász–type for hypergraphs with large girth are of special interest. In 1990 Szabó proved (see [22]) the following lower bound for $W(n,2)$: for any $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that for all $n \geqslant n_0$,

$$W(n,2) \geqslant 2^n n^{-\varepsilon}. \tag{10}$$

Szabó's approach was to consider the hypergraph of arithmetic progressions $H_n(N)$ as an "almost simple" hypergraph. He noticed that the fraction of sizes of edge pairwise intersections that are not equal to 0 or 1 is sufficiently small. It is easy to see that for any arithmetic progression A of length n, there is at most n^4 other progressions sharing more than one integer with A. On the other hand the number of edges of $H_n(N)$ intersecting A is $\Omega(N)$. Based on these simple observations Szabó applied the probabilistic technique used for colorings of simple hypergraphs to the hypergraph of arithmetic progressions. Although $H_n(N)$ is not simple, such small number of 2-cycles allowed the recoloring procedure to succeed for it also.

The lower bound (10) obviously improves (9). Moreover, since $\varepsilon > 0$ is arbitrary in the right-hand side of (10), the bound, actually, is of the form

$$W(n,2) \geqslant 2^n n^{-\varepsilon(n)}, \tag{11}$$

where $\varepsilon(n) \to 0$ as $n \to \infty$. The calculations from Szabó's proof imply that $\varepsilon(n)$ has an order $1/\sqrt[4]{\ln n}$. Later the estimate (11) was improved by the second author of this paper in [20], where it was shown that one can take $\varepsilon(n) = \Theta\left(\sqrt{\frac{\ln\ln n}{\ln n}}\right)$. The proof of develops Szabó's approach and considers the hypergraph of arithmetic progressions $H_n(N)$ as a hypergraph almost without 2- and 3-cycles.

The second main result of the paper provides a new lower bound for the Van der Waerden number.

Theorem 3. *There exists an absolute constant $c > 0$ such that, for any $n \geqslant 9$, $r \geqslant 2$,*

$$W(n,r) \geqslant c \cdot r^{n-1} \frac{(\ln \ln n)^2}{\ln n}. \tag{12}$$

The result of Theorem 3 obviously improves the previous best bound. In the proof we go further in the development of Szabó's approach and consider $H_n(N)$ as a hypergraph almost without short cycles.

When the color parameter r is large in comparison with n, one can combine Hypergraph symmetry theorem (the reader is referred to the monograph [8] for the details) and the lower bounds for the function $r_n(N)$ (see, e.g., [14]) to obtain better estimates than given above. Particularly, for fixed n and large r, the following inequality can be deduced

$$W(n,r) \geqslant \exp\left\{(\ln r)^m (1 + \psi_n(r))\right\}, \tag{13}$$

where $m = \lceil \log_2 n \rceil$ and $\psi_n(r) \to 0$ as $r \to \infty$. However, the bound (13) becomes nontrivial only if r is very large in comparison with n: $r \geqslant n^{\Omega(\sqrt{n})}$. Note that for $n = 3$ (13) coincides with the old result of Moser (6).

So, we established a new lower bound (12) for the Van der Waerden number in the wide area of the values of the parameters:

$$\ln r = O(\sqrt{n} \ln n).$$

The structure of the remaining part of the paper will be the following. Our proof of Theorem 2 is a reduction argument from the similar statement concerning colorings of uniform hypergraphs with girth at least 5.

Theorem 4. *Suppose $n \geqslant 9$, $r \geqslant 2$ and H is an n-uniform hypergraph with $g(H) > 5$. There exists an absolute constant $c > 0$ such that if*

$$\Delta(H) \leqslant c \cdot r^{n-1} \frac{n(\ln \ln n)^2}{\ln n} \tag{14}$$

then H is r-colorable.

So, in the next section we shall prove the main Theorem 4. Theorem 3 concerning the Van der Waerden number is derived in Section 4. Finally, in Section 5 we shall deduce Theorem 2.

3 Proof of Theorem 4

To prove that the hypergraph H is r-colorable we have to show the existence of a proper vertex r-coloring for it. We shall construct some random r-coloring and estimate the probability that this coloring is not proper for H.

3.1 Construction of random coloring

The construction of a random coloring is based on the method of *random recoloring*. This technique was introduced by J. Beck (see [2]) and then developed by J. Spencer (see [21]), Radhakrishnan and Srinivasan (see [15]) for two–colorings. In our work we use the ideas of Radhakrishnan and Srinivasan [15] concerning colorings of sparse hypergraphs.

Without loss of generality, we assume that $V = \{1, \ldots, m\}$. The construction consists of two stages.

First stage. Initial coloring At this stage we randomly color the vertices of the hypergraph with r colors uniformly and independently. Namely, we consider $\xi_1, \ldots, \xi_m$ — equally distributed independent random variables taking values $1, 2, \ldots, r$ with equal probability $1/r$. Random vector $\xi = (\xi_1, \ldots, \xi_m)$ can be interpreted as a random r-coloring of the vertex set V (we assign the color ξ_i to the vertex i).

The random coloring ξ can contain monochromatic and almost monochromatic edges. We say that an edge $A \in E$ is *almost monochromatic* in ξ if there is a color $u \in \{1, \ldots, r\}$ such that the number of vertices in A which are colored with u in ξ is at least $n - s$ and at most $n - 1$. In this case the color u is called *dominating* in A. Here $s < n/2$ is a parameter of our construction. In our proof we will set $s \sim \ln n$, so s will be small in comparison with n.

Formally, for any $A \in E$ and every $u = 1, \ldots, r$, let $\mathcal{M}(A, u)$ and $\mathcal{AM}(A, u)$ denote the following events:

$$\mathcal{M}(A, u) = \bigcap_{j \in A} \{\xi_j = u\}, \quad \mathcal{AM}(A, u) = \left\{ 1 \leqslant \sum_{j \in A} I\{\xi_j \neq u\} \leqslant s \right\}. \tag{15}$$

Here $I\{\mathcal{B}\}$ denotes an indicator of the event $\mathcal{B}$. It is clear that $\mathcal{M}(A, u)$ denotes the event that A is monochromatic of a color u in ξ, and $\mathcal{AM}(A, u)$ denotes the event that A is almost monochromatic with dominating color u in ξ.

Second stage. Process of random recoloring. The main principle of the random recoloring method is very clear: during the recoloring stage we would like to recolor some vertices from the monochromatic edges to make them non-monochromatic. The crucial idea provided by Radhakrishnan and Srinivasan is to pay special attention to the almost monochromatic edges during the recoloring procedure. Since they are very close to be monochromatic, we will forbid them to become completely monochromatic of dominating color.

To make a formal construction of the described idea consider the following set of random variables:

1. $X_1, \ldots, X_m$ — independent identically distributed random variables (also independent with ξ) with uniform distribution on $[0, 1]$, i.e. for any $j = 1, \ldots, m$,
$$\mathsf{Pr}\,(X_j < x) = x, \ x \in [0, 1].$$
2. $\{\eta_1, \ldots, \eta_m\}$ — independent identically distributed random variables (also independent with $X_1, \ldots, X_m$) taking values $1, 2, \ldots, r$ with the following conditional distribution: for every $j = 1, \ldots, m$,
$$\mathsf{Pr}\,(\eta_j = u | \xi_j = a) = \frac{1}{r-1} \text{ for any } u \neq a \in \{1, \ldots, r\},$$

i.e. η_j has uniform conditional distribution on the set $\{1, \ldots, r\} \setminus \{\xi_j\}$.

A continuous–time process of random recoloring goes as follows. Every vertex $v \in V$ is considered only at the time X_v and at this moment of the procedure we check the following two conditions:

Cond1 There is an edge A, $v \in A$, which is monochromatic in the coloring ξ and none of the vertices of A changed the initial color up to time X_v.

Cond2 The recoloring with color η_v is not blocked. We say that the recoloring of the vertex v with a color u is *blocked*, if there is an edge B, $v \in B$, such that B is almost monochromatic with dominating color u in ξ and at the time moment X_v the vertex v is the last vertex in B which is not colored with u.

If both conditions **Cond1** and **Cond2** hold then we recolor v with color η_v. Otherwise, we do not recolor v and the process continues.

For a vertex v and a time moment $t \geqslant 0$ let us define a random variable $\zeta_v(t)$, corresponding to the color of v at the time t:

$$\zeta_v(t) = \begin{cases} \xi_v, & \text{if } t < X_v, \\ \xi_v, & \text{if } t \geqslant X_v \text{ and one of the conditions } \mathbf{Cond1}, \mathbf{Cond2} \text{ does not hold}, \\ \eta_v, & \text{if } t \geqslant X_v \text{ and both } \mathbf{Cond1}, \mathbf{Cond2} \text{ hold}. \end{cases}$$

Thus, for any $t \geqslant 0$, we have the random r-coloring $\zeta(t) = (\zeta_1(t), \ldots, \zeta_m(t))$ for hypergraph H. Our aim is to show that for some $t \in (0, 1)$ and s, the coloring $\zeta(t)$ is a proper r-coloring for hypergraph H under the conditions of Theorem 4.

Remark 1. In fact we need only a random ordering of the vertex set to start the recoloring procedure. In our construction we order the vertices according to the values of the random variables $X_1, \ldots, X_m$. Nevertheless, using continuous time usually helps to simplify the calculations (we have to deal with integrals instead of discrete sums).

3.2 Bad events

Consider the situation that the coloring $\zeta(t)$, $t > 0$, is not proper for H. Let us denote this event by $\mathcal{F}(t)$. Suppose an edge $A \in E$ is monochromatic in $\zeta(t)$. We have the following partition of $\mathcal{F}(t)$ into three different classes of events.

1. Bad event of the first type occurs when there is an edge A satisfying the following conditions:

    - A is monochromatic in the initial coloring ξ,
    - A is still monochromatic of the same color in $\zeta(t)$,
    - up to the time t we have already considered at least h vertices of A (h is another parameter, we will choose $h \sim \ln n / \ln \ln n$).

Let us denote by $\mathcal{E}(A,t)$ the described bad event.

2. Bad event of the second type (denoted by $\mathcal{D}(A,t)$) occurs when there is an edge A satisfying the following conditions:

   - A is monochromatic in ξ,
   - it is still monochromatic of the same color in $\zeta(t)$,
   - up to the time t we have considered at most $h-1$ vertices of A.

3. Bad event of the third type happens if there is an edge A such that

   - A is monochromatic of a color u in the coloring $\zeta(t)$,
   - it is not monochromatic of u in the initial coloring ξ.

   The last bad event is denoted by $\mathcal{C}(A,t)$.

It is easy to see that $\mathcal{F}(t)$ is a union of the introduced bad events:

$$\mathcal{F}(t) = \bigcup_{A\in E} \left(\mathcal{E}(A,t) \cup \mathcal{D}(A,t) \cup \mathcal{G}(A,t)\right). \tag{16}$$

In the next few paragraphs we shall analyze the bad events more closely.

3.3 First bad event

Suppose that the event $\mathcal{E}(A,t)$ occurs for some edge $A \in E$. This event implies that the edge A is monochromatic in the initial coloring and after the consideration of its first h vertices (according to the ordering provided by the random variables $X_1,\ldots,X_m$) it is still monochromatic of the same color, no recoloring has been made. We will show that a special hypertree configuration of depth at most 3 is responsible for this.

Suppose $v_1,\ldots,v_h$ are the first h vertices of A to be considered, i.e.

$$X_{v_1} < X_{v_2} < \ldots < X_{v_h} \text{ and}$$

$$X_{v_h} < X_{v'},\ \text{for any } v' \in A\setminus\{v_1,\ldots,v_h\}.$$

At the time X_{v_h} we have considered all of them, but no recoloring has been made. Why did we not change their colors? Since A is monochromatic in the initial coloring ξ, the condition **Cond1** holds for every v_i. So, the second condition **Cond2** does not hold, i.e. the recoloring of v_i with color η_{v_i} is blocked by some almost monochromatic edge B_i. Due to our algorithm we have the following properties for the *blocking* edge B_i:

BL1 B_i is almost monochromatic in the initial coloring ξ with dominating color η_{v_i},

BL2 at the time X_{v_i} the vertex v_i remains the only one vertex of B_i, which is not colored with η_{v_i}.

Property **BL2** implies that any other vertex of B_i, which is not colored with η_{v_i} in the coloring ξ should be recolored with η_{v_i} up to the time X_{v_i}. And the first property **BL1** says that the number of such vertices is at most $s-1$ (recall that v_i is not colored with η_{v_i} in ξ).

Let $\{w_{i,1}\dots,w_{i,y_i}\}$, where $y_i \leqslant s-1$, denote this subset of vertices from $B_i \setminus \{v_i\}$. Since every vertex $w_{i,j}$ ($i=1,\dots,h$, $j=1,\dots,y_i$) has been recolored with color η_{v_i} up to the time X_{v_i}, we have $\eta_{w_{i,j}} = \eta_{v_i}$ and, moreover, there is an edge $C_{i,j}$ containing $w_{i,j}$ such that

- $C_{i,j}$ is monochromatic in the initial coloring ξ,
- up to the time $X_{w_{i,j}}$ no recoloring has been made in the edge $C_{i,j}$, i.e. $w_{i,j}$ is the first vertex of $C_{i,j}$, which changes its color during the recoloring process.

Thus, we get a hypertree with "trunk" A, "branches" $B_1,\dots,B_h$ and "leaves" $C_{1,1},\dots,C_{1,y_1}$, $C_{2,1},\dots,C_{h,1},\dots,C_{h,y_h}$ ($C_{i,1},\dots,C_{i,y_i}$ are the "leaves" of "branch" B_i). Since, girth of hypergraph H is greater than 5, it is really a hypertree.

Let us sum up our analysis. We define *a first–type configuration* $(A,\Phi,\overrightarrow{y},\Lambda)$ as follows:

- A is an edge of hypergraph H;
- $\Phi=(B_1,\dots,B_h)$ is an ordered collection of distinct edges of A such that $|B_i\cap A|=1$ for any i, and the vertices $v_i = B_i\cap A$ are distinct;
- $\overrightarrow{y}=(y_1,\dots,y_h)$ is a vector from $\{0,1,\dots,s-1\}^h$;
- Λ is an unordered collection of edges $\Lambda=\{C_{1,1},\dots,C_{1,y_1},C_{2,1},\dots,C_{h,y_h}\}$ such that $|C_{i,j}\cap B_i|=1$ for any i,j, and all the vertices $w_{i,j}=C_{i,j}\cap B_i$ are distinct;
- the set of edges A, $B_1,\dots,B_h$, $C_{1,1},\dots,C_{h,y_h}$ forms a hypertree with "trunk" A, "branches" $B_1,\dots,B_h$ and "leaves" $C_{1,1},\dots,C_{h,y_h}$.

The set of first–type configurations we shall denote by CONF1. The above discussion shows that the event $\mathcal{E}(A,t)$ implies an event $\mathcal{A}_0(A,\Phi,\overrightarrow{y},\Lambda)$ for some first–type configuration $(A,\Phi,\overrightarrow{y},\Lambda)$, where there the event $\mathcal{A}_0(A,\Phi,\overrightarrow{y},\Lambda)$ means that

1. A is monochromatic in ξ; no recoloring is made up to the consideration of its first h vertices $v_1,\dots,v_h$;
2. every B_i is almost monochromatic in ξ; v_1, $w_{i,1},\dots,w_{i,y_i}$ are the vertices which are not colored with dominating color); at the time X_{v_i} only v_i is not colored with the dominating color;
3. every $C_{i,j}$ is monochromatic in ξ; $w_{i,j}$ is its first recolored vertex during the recoloring procedure.

Thus, we have
$$\mathcal{E}(A,t)\subset \bigcup_{\substack{\Phi,\overrightarrow{y},\Lambda:\\ (A,\Phi,\overrightarrow{y},\Lambda)\in \text{CONF1}}} \mathcal{A}_0(A,\Phi,\overrightarrow{y},\Lambda). \tag{17}$$

Now we are going to analyze the event $\mathcal{A}_0(A, \Phi, \overrightarrow{y}, \Lambda)$ more closely. Since this event states that any leaf–edge $C_{i,j} \in \Lambda$ is monochromatic, the recoloring of any vertex preceding $w_{i,j}$ in this edge is blocked (recall that $w_{i,j}$ is the first vertex of $C_{i,j}$, which changes its color during the recoloring process.). The number of such vertices is determined by a random variable

$$\sum_{w' \in C_{i,j} \setminus \{w_{i,j}\}} I\{X_{w'} < X_{w_{i,j}}\}.$$

Let $\mathcal{A}(A, \Phi, \overrightarrow{y}, \Lambda)$ denote the event $\mathcal{A}_0(A, \Phi, \overrightarrow{y}, \Lambda)$ with additional condition that this number of vertices does not exceed $h-1$ for any i and j, i.e. in any leaf–edge from Λ at most $h-1$ vertices had been considered before the successful recoloring was made. The following claim is a key for the analysis of the event $\mathcal{A}_0(A, \Phi, \overrightarrow{y}, \Lambda)$.

Claim 1. *The event $\mathcal{A}_0(A, \Phi, \overrightarrow{y}, \Lambda)$ satisfies the relation*

$$\mathcal{A}_0(A, \Phi, \overrightarrow{y}, \Lambda) \subset \bigcup_{\substack{A', \Phi', \overrightarrow{y}', \Lambda': \\ (A', \Phi', \overrightarrow{y}', \Lambda') \in \text{CONF1}}} \mathcal{A}(A', \Phi', \overrightarrow{y}', \Lambda'). \tag{18}$$

Proof. For any edge $A \in E$, we introduce the random variable $Z(A)$ equal to the value of X_v, where a vertex $v \in A$ is the h-th vertex in A to be considered during the recoloring process, i.e.

$$\sum_{j \in A} I\{X_j < X_v\} = h - 1.$$

Now, suppose the event $\mathcal{A}_0(A, \Phi, \overrightarrow{y}, \Lambda)$ holds and for some leaf–edge $C_{i,j} \in \Lambda$ we have

$$\sum_{w' \in C_{i,j} \setminus \{w_{i,j}\}} I\{X_{w'} < X_{w_{i,j}}\} \geqslant h. \tag{19}$$

Hence, during the recoloring process we considered first h vertices in the monochromatic edge $C_{i,j}$ and no recoloring was made. Consequently, for some first–type configuration $(C_{i,j}, \widetilde{\Phi}, \overrightarrow{\widetilde{y}}, \widetilde{\Lambda})$, the event $\mathcal{A}_0(C_{i,j}, \widetilde{\Phi}, \overrightarrow{\widetilde{y}}, \widetilde{\Lambda})$ occurs. It is easy to see from (19) that $Z(C_{i,j}) < Z(A)$ since

$$Z(C_{i,j}) < X_{w_{i,j}} < X_{v_i} \leqslant X_{v_h} = Z(A).$$

If none of the leaf–edges $\widetilde{C}_{l,k} \in \widetilde{\Lambda}$ satisfies the condition

$$\sum_{w' \in \widetilde{C}_{l,k} \setminus \{\widetilde{w}_{l,k}\}} I\{X_{w'} < X_{\widetilde{w}_{l,k}}\} \geqslant h,$$

then the event $\mathcal{A}(C_{i,j}, \widetilde{\Phi}, \overrightarrow{\widetilde{y}}, \widetilde{\Lambda})$ holds and we are done. Otherwise, we can replace edge A by $C_{i,j}$ in the above argument and apply it to the event $\mathcal{A}_0(C_{i,j}, \widetilde{\Phi}, \overrightarrow{\widetilde{y}}, \widetilde{\Lambda})$.

Since our hypergraph is finite and the value of the function Z decreases, after several repetitions of the described argument we will get an edge A' and a first–type configuration $(A', \Phi', \overrightarrow{y}', \Lambda')$ such that in every monochromatic leaf–edge from Λ' at most $h-1$ vertices had been considered before the successful recoloring was made (note that the set Λ' can be empty). This implies the event $\mathcal{A}(A', \Phi', \overrightarrow{y}', \Lambda')$ and the relation (18) is established. $\square$

It remains to estimate the probability of the event $\mathcal{A}(A, \Phi, \overrightarrow{y}, \Lambda)$ for given first–type configuration $(A, \Phi, \overrightarrow{y}, \Lambda)$.

Claim 2.

$$\mathsf{Pr}\left(\mathcal{A}(A,\Phi,\overrightarrow{y},\Lambda)\right) \leqslant r^{-(n-1)(1+h+\sum_{i=1}^h y_i)} \left(\frac{h}{n}\right)^{\sum_{i=1}^h y_i} (n-h)^{-h}. \quad (20)$$

Proof. Let us fix a color u of an edge A in ξ, dominating colors $u_1,\ldots,u_h$ for the edges $B_1,\ldots,B_h$ and the colors $u_{i,j}$ for monochromatic edges $C_{i,j}$. Our construction of the hypertree implies that, for given values of u, u_i and $u_{i,j}$, $i \leqslant h$, $j \leqslant y_i$, the initial of all the vertices in the configuration are uniquely determined:

$$\xi_v = \begin{cases} u, & \text{if } v \in A; \\ u_i, & \text{if } v \in B_i \setminus (A \cup \bigcup_{j=1}^{y_i} C_{i,j}); \\ u_{i,j}, & , \text{ if } v \in C_{i,j}. \end{cases}$$

The number of edges in the hypertree is equal to $1+h+\sum_{i=1}^h y_i$, so the number of vertices is equal to $1+(n-1)(1+h+\sum_{i=1}^h y_i)$ and there should be a factor $r^{-1-(n-1)(1+h+\sum_{i=1}^h y_i)}$.

The values of the random variables η are also uniquely determined for the node vertices of the hypertree:

$$\eta_v = \begin{cases} u_i, & \text{if } v = v_i,\ i = 1,\ldots,h; \\ u_i, & \text{if } v = w_{i,j},\ i = 1,\ldots,h,\ j = 1,\ldots,y_i. \end{cases}$$

Hence, we have a factor $(r-1)^{-(h+\sum_{i=1}^h y_i)}$.

Furthermore, we know that the number of every vertex $w_{i,j}$ in the edge $C_{i,j}$ is at most h, all these events are independent, so we have a factor $(h/n)^{\sum_{i=1}^h y_i}$.

Finally, we know that v_i has the number i in the edge A, $i = 1,\ldots,h$. So, the last factor is $[n(n-1)\ldots(n-h+1)]^{-1}$.

Gathering together the above arguments, we obtain that

$$\mathsf{Pr}\left(\mathcal{A}(A,\Phi,\overrightarrow{y},\Lambda)\right) = \sum_{u=1}^{r} \sum_{\substack{u_1,\ldots,u_h=1,\\ u_i\neq u,\ i=1,\ldots,h}}^{r} \sum_{\substack{u_{1,i},\ldots,u_{h,y_h}=1,\\ u_{i,j}\neq u_i,\ j=1,\ldots,y_i}}^{r} r^{-1-(n-1)(1+h+\sum_{i=1}^h y_i)}(r-1)^{-(h+\sum_{i=1}^h y_i)} \times$$

$$\times \left(\frac{h}{n}\right)^{\sum_{i=1}^h y_i} \frac{1}{n(n-1)\ldots(n-h+1)} =$$

(the colors can be chosen in $r(r-1)^{h+\sum_{i=1}^h y_i}$ ways)

$$= r^{-(n-1)(1+h+\sum_{i=1}^h y_i)} \left(\frac{h}{n}\right)^{\sum_{i=1}^h y_i} \frac{(n-h)!}{n!} \leqslant r^{-(n-1)(1+h+\sum_{i=1}^h y_i)} \left(\frac{h}{n}\right)^{\sum_{i=1}^h y_i} (n-h)^{-h}.$$

□

That is all with the first bad event and we proceed to the second one.

3.4 Second bad event

Let us consider the bad event $\mathcal{D}(A,t)$ of the second type. It is obvious that this event can happen only if we have considered less that h vertices, i.e. $\sum_{v\in A} I\{X_v \leqslant t\} \leqslant h-1$. We know also that A is monochromatic in ξ. Denoting the intersection of these events by $\mathcal{B}(A,t)$, we get

$$\mathcal{D}(A,t) \subset \mathcal{B}(A,t). \tag{21}$$

The random variables X_v are independent of the initial coloring ξ, thus for $t > 2/n$, we obtain the following upper bound for the probability of the event $\mathcal{B}(A,t)$:

$$\mathsf{Pr}\left(\mathcal{B}(A,t)\right) = r^{1-n}\sum_{i=0}^{h-1}\binom{n}{i}t^i(1-t)^{n-i} \leqslant r^{1-n}(1-t)^{n-h+1}\sum_{i=0}^{h-1}(nt)^i \leqslant$$

$$\leqslant r^{1-n}(1-t)^{n-h+1}(nt)^h. \tag{22}$$

3.5 Third event.

The last part of the event $\mathcal{F}(t)$ is a bad event $\mathcal{G}(A,t)$ of the third type. Suppose that the edge A is monochromatic of a color u in the coloring $\zeta(t)$, but in the initial coloring ξ it was not monochromatic of this color. Since during the recoloring process we forbid almost monochromatic edges to become completely monochromatic of a dominating color, A is not almost monochromatic with dominating coloring u in ξ. Hence, the number of vertices in A which are not colored with u in the coloring ξ is at least $s+1$ (see (15)).

Suppose that $\{v_1,\dots,v_l\}$ is the set of all vertices of A which are not colored with u in ξ. Since $\zeta_{v_i}(t) = u$ for any $i = 1,\dots,l$, all these vertices should be recolored with u up to the time t. Thus, for any v_i, at the time X_{v_i} both conditions **Cond1** and **Cond2** hold. In this case our construction provides the existence of the set of l edges, $\{B_1,\dots,B_l\}$ with the following properties:

1. $v_i \in B_i \cap A$ and $\eta_{v_i} = u$, $i = 1,\dots,l$;

2. B_i is monochromatic in the initial coloring ξ of other color than u;

3. v_i is the first recolored vertex in B_i during the recoloring process.

Moreover, we can assume that for every edge B_i, the event $\mathcal{E}(A,t)$ does not hold, since we have already analyzed it in section 3.3. The completion of this event implies that the number of vertices considered in B_i before v_i is at most $h-1$, otherwise the edge B_i would be still monochromatic after the consideration of its h-th vertex. Hence, there is one more property of our configuration of edges.

4. for any $i = 1,\dots,l$,

$$\sum_{v\in B_i\setminus\{v_i\}} I\{X_v < X_{v_i}\} \leqslant h-1.$$

Since the girth of hypergraph H is at least 6, the set of edges $A, B_1, \dots, B_l$ forms a hypertree with "trunk" A and "branches" $B_1, \dots, B_l$. Note that the edges $B_1, \dots, B_l$ are pairwise disjoint since all the vertices $v_1, \dots, v_l$ are distinct.

Let us define *a second–type configuration* (A, l, Φ) as follows:

- A is an edge of E;
- $l \in \{s+1, \dots, n\}$;
- Φ is an unordered collection of distinct edges $\Phi = \{B_1, \dots, B_l\}$, such that $|B_i \cap e| = 1$ for any i, and the vertices $v_i = B_i \cap A$ are distinct;
- the set of edges $A, B_1, \dots, B_l$ forms a hypertree.

The set of all second–type configurations we shall denote by CONF2 and the described above event will be denoted by $\mathcal{C}(A, l, \Phi)$. The above discussion implies the following relation

$$\mathcal{G}(A,t) \cap \overline{\bigcup_{B \in E} \mathcal{E}(B,t)} \subset \bigcup_{l,\Phi:\ (A,l,\Phi) \in \text{ CONF2}} \mathcal{C}(A,l,\Phi). \tag{23}$$

Let us estimate the probability of the event $\mathcal{C}(A, l, \Phi)$ for a given second–type configuration (A, l, Φ).

Claim 3.

$$\mathsf{Pr}\left(\mathcal{C}(A,l,\Phi)\right) = r^{-(n-1)(l+1)} \left(\frac{h}{n}\right)^l. \tag{24}$$

Proof. The proof is almost obvious. Let us fix a color u as a color of A in the final coloring and $u_1, \dots, u_l$ as colors of $B_1, \dots, B_l$ in the initial coloring. Thus, for given colors, the colors of all the vertices in the configuration are uniquely defined:

$$\xi_v = \begin{cases} u, & \text{if } v \in A \setminus (\bigcup_{i=1}^l B_i); \\ u_i, & \text{if } v \in B_i. \end{cases}$$

The number of edges in the hypertree is equal to $1 + l$, so the number of vertices is equal to $1 + (n-1)(1+l)$ and there should be a factor $r^{-1-(n-1)(1+l)}$. The values of the random variables η are also uniquely determined for the vertices v_i: $\eta_{v_i} = u$. Hence, we have a factor $(r-1)^{-l}$. Finally, every vertex v_i has a number at most h in the edge B_i. So we have a factor $(h/n)^l$. Consequently,

$$\mathsf{Pr}\left(\mathcal{C}(A,l,\Phi)\right) = \sum_{u=1}^{r} \sum_{\substack{u_1,\dots,u_l=1, \\ u_i \neq u,\ i=1,\dots,l}}^{r} r^{-1-(n-1)(1+l)} (r-1)^{-l} \left(\frac{h}{n}\right)^l =$$

$$= r(r-1)^l r^{-1-(n-1)(1+l)} (r-1)^{-l} \left(\frac{h}{n}\right)^l = r^{-(n-1)(l+1)} \left(\frac{h}{n}\right)^l.$$

$\square$

3.6 Application of Local Lemma

In the previous paragraphs we have analyzed the bad events, the parts of the event $\mathcal{F}(t)$. Remember that $\mathcal{F}(t)$ is the event that the random coloring $\zeta(t)$ is not a proper r-coloring for hypergraph H. Recall the equality (16):

$$\mathcal{F}(t) = \bigcup_{A\in E} \left(\mathcal{E}(A,t) \cup \mathcal{D}(A,t) \cup \mathcal{G}(A,t)\right).$$

It follows from the relations (17), (18), (21) and (23) that

$$\mathcal{F}(t) \subset \bigcup_{(A,\Phi,\overrightarrow{y},\Lambda)\in \text{ CONF1}} \mathcal{A}(A,\Phi,\overrightarrow{y},\Lambda) \cup \bigcup_{A\in E} \mathcal{B}(A,t) \cup \bigcup_{(A,l,\Phi)\in \text{ CONF2}} \mathcal{C}(A,l,\Phi). \tag{25}$$

Our aim is to show that, for some parameters h, s and $t \in (0,1)$, the probability of the event $\mathcal{F}(t)$ is strictly less than 1. To prove this we shall use a classical statement, which is called Local Lemma. This statement was first obtained in the paper of P. Erdős and L. Lovász [5]. We shall formulate it in a special case convenient for later use.

Theorem 5. *Let events $\mathcal{Q}_1, \ldots, \mathcal{Q}_M$ be given on some probability space. Let $S_1, \ldots, S_M$ be subsets of $\mathcal{R}_M = \{1, \ldots, M\}$ such that for any $i = 1, \ldots, M$, the event $\mathcal{Q}_i$ is independent of the algebra generated by the events $\{\mathcal{Q}_j, j \in \mathcal{R}_M \backslash S_i \cup \{\mathcal{Q}_i\}\}$. If, for any $i = 1, \ldots, M$, the following inequality holds*

$$\sum_{j\in S_i \cup\{i\}} \mathsf{Pr}(\mathcal{Q}_j) \leqslant 1/4, \tag{26}$$

then $\mathsf{Pr}\left(\bigcap_{j=1}^{M} \overline{\mathcal{Q}_j}\right) \geqslant \prod_{j=1}^{M} (1 - 2\mathsf{Pr}(\mathcal{Q}_j)) > 0.$

The proof of the Local Lemma in the general case can be found in the monograph [1].

Consider the system of events $\Psi(t)$ consisting of all the events $\mathcal{B}(A,t)$, $A \in E$, all the events $\mathcal{A}(A,\Phi,\overrightarrow{y},\Lambda)$, $(A,\Phi,\overrightarrow{y},\Lambda) \in \text{CONF1}$, and all the events $\mathcal{C}(A,l,\Phi)$, $(A,l,\Phi) \in \text{CONF2}$. By (25) we have

$$\mathsf{Pr}\left(\mathcal{F}(t)\right) \leqslant \mathsf{Pr}\left(\bigcup_{\mathcal{Q}\in \Psi(t)} \mathcal{Q}\right) = 1 - \mathsf{Pr}\left(\bigcap_{\mathcal{Q}\in \Psi(t)} \overline{\mathcal{Q}}\right). \tag{27}$$

We would like to show that the probability of $\bigcap_{\mathcal{Q}\in \Psi(t)} \overline{\mathcal{Q}}$ is greater than zero. Due to the Local Lemma it is sufficient to find, for every $\mathcal{Q} \in \Psi(t)$, a system of events $\Psi_{\mathcal{Q}} \subset \Psi(t)$ such that $\mathcal{Q} \in \Psi_{\mathcal{Q}}$, $\mathcal{Q}$ and the algebra generated by $\{\mathcal{J} \in \Psi(t)\backslash\Psi_{\mathcal{Q}}\}$ are independent, and, moreover, such that

$$\sum_{\mathcal{J}\in \Psi_{\mathcal{Q}}} \mathsf{Pr}(\mathcal{J}) \leqslant 1/4. \tag{28}$$

The event $\mathcal{Q} \in \Psi(t)$ can have one of the following three types:

1. $\mathcal{Q} = \mathcal{B}(A,t)$ for some $A \in E$;

2. $\mathcal{Q} = \mathcal{A}(A, \Phi, \overrightarrow{y}, \Lambda)$ for some first–type configuration $(A, \Phi, \overrightarrow{y}, \Lambda) \in$ CONF1;

3. $\mathcal{Q} = \mathcal{C}(A, l, \Phi)$ for some second-type configuration $(A, l, \Phi) \in$ CONF2.

For any $\mathcal{Q} \in \Psi(t)$, we define *the domain* $D(\mathcal{Q})$ of the event $\mathcal{Q}$ and the edge set $E(\mathcal{Q})$ as follows:

$$D(\mathcal{Q}) = \begin{cases} A, & \text{if } \mathcal{Q} = \mathcal{B}(A, t); \\ A \cup \bigcup\limits_{B \in \Phi} B \cup \bigcup\limits_{C \in \Lambda} C, & \text{if } \mathcal{Q} = \mathcal{A}(A, \Phi, \overrightarrow{y}, \Lambda); \\ A \cup \bigcup\limits_{B \in \Phi} B, & \text{if } \mathcal{Q} = \mathcal{C}(A, l, \Phi), \end{cases}$$

$$E(\mathcal{Q}) = \begin{cases} \{A\}, & \text{if } \mathcal{Q} = \mathcal{B}(A, t); \\ \{A\} \cup \Phi \cup \Lambda, & \text{if } \mathcal{Q} = \mathcal{A}(A, \Phi, \overrightarrow{y}, \Lambda); \\ \{A\} \cup \Phi, & \text{if } \mathcal{Q} = \mathcal{C}(A, l, \Phi). \end{cases}$$

By the definitions of the events $\mathcal{A}(A, \Phi, \overrightarrow{y}, \Lambda, \mathcal{B}(A, t))$ and $\mathcal{C}(A, l, \Phi)$ any $\mathcal{Q} \in \Psi(t)$ belongs to the algebra generated by the random variables $\{\xi_j, \eta_j, X_j : \ j \in D(\mathcal{Q})\}$. Then this event is independent of the algebra generated by the random variables $\{\xi_j, \eta_j, X_j : \ j \in V \setminus D(\mathcal{Q})\}$. So, we can take the system $\Psi_{\mathcal{Q}}$ consisting of all the events $\mathcal{J} \in \Psi(t)$ such that $D(\mathcal{J}) \cap D(\mathcal{Q}) \neq \emptyset$,

$$\Psi_{\mathcal{Q}} = \{\mathcal{J} : \ \mathcal{J} \in \Psi(t), \ D(\mathcal{J}) \cap D(\mathcal{Q}) \neq \emptyset\}.$$

Thus, the event $\mathcal{Q}$ would be independent of the algebra generated by $\{\mathcal{J} \in \Psi(t) \setminus \Psi_{\mathcal{Q}}\}$. Moreover, $\mathcal{Q} \in \Psi_{\mathcal{Q}}$. It remains to check the inequality (28). By the choice of the set $\Psi_{\mathcal{Q}}$ for any $\mathcal{Q} \in \Psi(t)$ we have

$$\sum_{\mathcal{J} \in \Psi_{\mathcal{Q}}} \mathsf{Pr}\,(\mathcal{J}) \leqslant \sum_{J \in E(\mathcal{Q})} \left[\sum_{A \in E\colon J \cap A \neq \emptyset} \mathsf{Pr}\,(\mathcal{B}(A, t)) + \sum_{\substack{(A, \Phi, \overrightarrow{y}, \Lambda) \in \text{ CONF1:} \\ J \cap \left(A \cup \bigcup\limits_{B \in \Phi} B \cup \bigcup\limits_{C \in \Lambda} C \right) \neq \emptyset}} \mathsf{Pr}\,(\mathcal{A}(A, \Phi, \overrightarrow{y}, \Lambda)) + \right.$$

$$\left. + \sum_{\substack{(A, l, \Phi) \in \text{ CONF2:} \\ J \cap \left(A \cup \bigcup\limits_{B \in \Phi} B \right) \neq \emptyset}} \mathsf{Pr}\,(\mathcal{C}(A, l, \Phi)) \right]. \tag{29}$$

Further we shall make analysis of the sums in the right-hand side of (29) separately. For convenience, we also use the notation Δ for the maximum edge degree $\Delta(H)$ of the hypergraph H. In fact, all we need to do is to to estimate the number of different configurations intersecting a fixed edge A.

For the first sum in the right-hand side of (29), the configuration consists of only one edge, so their number is at most $\Delta + 1$. So, we have to deal with the rest two big sums. Let us start with the second one.

Claim 4. *For a fixed edge J and $\overrightarrow{y} = (y_1, \ldots, y_h)$, there is a most*

$$\Delta^{1+h+\sum_{i=1}^{h} y_i} \left(\frac{1+hs}{\prod_{i=1}^{h}(y_i!)} \right)$$

configurations of the first type $(A,\ \Phi,\ \overrightarrow{y},\ \Lambda)$ intersecting J.

Proof. If $J \cap \left(A \cup \bigcup_{B\in\Phi} B \cup \bigcup_{C\in\Lambda} C\right) \neq \emptyset$ then there are three possibilities:

- Suppose $J \cap A \neq \emptyset$. Then edge A can be chosen in at most $\Delta+1$ ways, the ordered set Φ of h edges $\Phi = (B_1, \ldots, B_h)$ can be chosen in at most $\Delta(\Delta-1)\ldots(\Delta-h+1)$ ways (every B_i should intersect A and not coincide with it) and the unordered set Λ of $y_1 + \ldots + y_h$ edges $\Lambda = (C_{1,1}, \ldots, C_{1,y_1}, \ldots, C_{h,y_h})$ can be chosen in at most $\binom{\Delta-1}{x_1}\ldots\binom{\Delta-1}{x_h}$ ways (for any $i = 1, \ldots, h$, $j = 1, \ldots, y_h$, the edge $C_{i,j}$ should intersect B_i and not coincide with it and with A). Thus, the number of such configurations is at most

$$(\Delta+1)\Delta(\Delta-1)\ldots(\Delta-h+1)\left(\prod_{i=1}^{h}\binom{\Delta-1}{y_i}\right). \tag{30}$$

- Suppose $J \cap B_i \neq \emptyset$ for some $i = 1, \ldots, h$. Thus, the edge B_i and its number can be chosen in at most $h(\Delta+1)$ ways, the edge A can be chosen in at most $\Delta - 1$ ways (A should intersect B_i and not coincide with it and with J), the remained set of $h-1$ edges $(B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_h)$ can be chosen in at most $(\Delta-1)\ldots(\Delta-h+1)$ ways and, finally, the unordered set Λ of $y_1 + \ldots + y_h$ edges as in the previous case can be chosen by at most $\binom{\Delta-1}{y_1}\ldots\binom{\Delta-1}{y_h}$ ways. Thus, the number of configurations in the second situation is at most

$$h(\Delta+1)(\Delta-1)(\Delta-1)\ldots(\Delta-h+1)\left(\prod_{i=1}^{h}\binom{\Delta-1}{y_i}\right). \tag{31}$$

- The last variant is that $J \cap C_{i,j} \neq \emptyset$ for some $i = 1, \ldots, h$, $j = 1, \ldots, y_i$. In this case the edge $C_{i,j}$ can be chosen in at most $\Delta+1$ ways, the edge B_i can be chosen in at most $\Delta-1$ ways (B_i should intersect $C_{i,j}$ and not coincide with it and with J), the edge A can be chosen by at most $\Delta-1$ ways (A should intersect B_i and not coincide with it and with $C_{i,j}$), the remained set of $h-1$ edges $(B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_h)$ can be chosen in at most $(\Delta-1)\ldots(\Delta-h+1)$ ways and the rest of the set Λ, i.e. $(C_{1,1}, \ldots, C_{h,y_h})$ without $C_{i,j}$, can be chosen by at most $\binom{\Delta-1}{y_1}\ldots\binom{\Delta-2}{y_i-1}\ldots\binom{\Delta-1}{y_h}$ ways. Thus, in this case the number of configurations is at most

$$\sum_{i=1}^{h}(\Delta+1)(\Delta-1)^2(\Delta-1)\ldots(\Delta-h+1)\left(\prod_{i=1}^{h}\binom{\Delta-1}{y_i}\right)\frac{y_i}{\Delta-1} =$$

$$= (\Delta+1)(\Delta-1)(\Delta-1)\ldots(\Delta-h+1)\left(\prod_{i=1}^{h}\binom{\Delta-1}{y_i}\right)(y_1+\ldots+y_h). \tag{32}$$

Summarizing (30), (31) and (32) we obtain the following upper bound for the number of configurations of the first type with fixed $\overrightarrow{y}$:

$$(\Delta+1)(\Delta-1)\dots(\Delta-h+1)\left(\prod_{i=1}^{h}\binom{\Delta-1}{y_i}\right)\left((h+\sum_{i=1}^{h}y_i)(\Delta-1)+\Delta\right)\leqslant$$

$$\leqslant \Delta^{1+h+\sum_{i=1}^{h}y_i}\left(\frac{1+h+\sum_{i=1}^{h}y_i}{\prod_{i=1}^{h}(y_i!)}\right)\leqslant |y_i\leqslant s-1|\leqslant \Delta^{1+h+\sum_{i=1}^{h}y_i}\left(\frac{1+hs}{\prod_{i=1}^{h}(y_i!)}\right).$$

□

Claim 5. *For a fixed edge J and $l\geqslant s$, there is a most*

$$\Delta^{1+l}\left(\frac{1+l}{l!}\right)$$

configurations of the second type (A,l,Φ) intersecting J.

Proof. If $J\cap A\neq\emptyset$ then A can be chosen in at most $\Delta+1$ ways and the unordered set of edges $\Phi=\{B_1,\dots,B_l\}$ — in at most $\binom{\Delta}{l}$ ways. If $J\cap B_i\neq\emptyset$ for some $B_i\in\Phi$, then B_i can be chosen in at most $\Delta+1$ ways, the edge A — in at most $\Delta-1$ ways and the remained edges of Φ — in at most $\binom{\Delta-1}{l-1}$ ways. So, the whole number of configurations does not exceed

$$(\Delta+1)\binom{\Delta}{l}+(\Delta+1)(\Delta-1)\binom{\Delta-1}{l-1}\leq\frac{(l+1)\Delta^{l+1}}{l!}.$$

□

Gathering the inequality (29) together with the estimates of probabilities (20), (22), (24) and Claims 4, 5 we obtain that

$$\sum_{\mathcal{J}\in\Psi_{\mathcal{Q}}}\mathsf{Pr}\left(\mathcal{J}\right)\leqslant|E(\mathcal{Q})|\Bigg((\Delta+1)r^{1-n}(1-t)^{n-h+1}(nt)^h+n\sum_{y_1,\dots,y_h=0}^{s-1}\left(\frac{h\Delta}{r^{n-1}n}\right)^{\sum_{i=1}^{h}y_i}\times$$

$$\times\left(\frac{\Delta}{r^{n-1}n}\right)^{h+1}\left(\frac{n}{n-h}\right)^{h}\frac{1+hs}{\prod_{i=1}^{h}(y_i!)}+\frac{n}{h}\sum_{l=s+1}^{n}\frac{(l+1)}{l!}\left(\frac{h\Delta}{n\,r^{n-1}}\right)^{l+1}\Bigg).\tag{33}$$

To complete the proof of the theorem we have to show that the right-hand side of (33) is at most 1/4 for some choice of the parameters.

3.7 Choice of parameters and the completion of the proof

The cardinality of the set $E(\mathcal{Q})$ can be easily calculated:

$$|E(\mathcal{Q})|=\begin{cases}1, & \text{if } \mathcal{Q}=\mathcal{B}(A,t);\\ 1+h+y_1+\ldots+y_h, & \text{if } \mathcal{Q}=\mathcal{A}(A,\Phi,\overrightarrow{y},\Lambda);\\ 1+l, & \text{if } \mathcal{Q}=\mathcal{C}(A,l,\Phi).\end{cases}\tag{34}$$

Let us make the following choice of the parameters s and h:

$$h = \left\lfloor \frac{3\ln n}{\ln\ln n} \right\rfloor, \quad s = \lfloor \ln n \rfloor. \tag{35}$$

Such selection of the parameters is correct: $s < n/2$ for any $n \geqslant 3$ and $h < n$ for any $n \geqslant 9$.

Since the parameter l does not exceed n and every y_i is at most $s-1$ (hence, $1 + h + y_1 + \ldots + y_h \leq 1 + h + h(s-1) \leqslant 4(\ln n)^2$), it is obvious from (34) that $|E(\mathcal{Q})| \leqslant n+1$ for any $\mathcal{Q} \in \Psi(t)$.

Using the initial restriction (14) on the maximum edge degree Δ of the hypergraph H (recall that $\Delta \leqslant c\, r^{n-1} n (\ln\ln n)^2/\ln n$) and our choice of the parameters (35), we get the following upper bounds for the second and the third summands of the right-hand side of (33):

$$n \sum_{y_1,\ldots,y_h=0}^{s-1} \left(\frac{h\Delta}{r^{n-1}n}\right)^{\sum_{i=1}^h y_i} \left(\frac{\Delta}{r^{n-1}n}\right)^{h+1} \left(\frac{n}{n-h}\right)^h \frac{1+hs}{\prod_{i=1}^h (y_i!)} \leqslant$$

$$\leqslant n \sum_{y_1,\ldots,y_h=0}^{s-1} (3c\ln\ln n)^{\sum_{i=1}^h y_i} \left(\frac{c(\ln\ln n)^2}{\ln n}\right)^{\frac{3\ln n}{\ln\ln n}} \left(\frac{n}{n-h}\right)^h \frac{4(\ln n)^2}{\prod_{i=1}^h (y_i!)} =$$

$$= n \left(\frac{c(\ln\ln n)^2}{\ln n}\right)^{\frac{3\ln n}{\ln\ln n}} \left(\frac{n}{n-h}\right)^h 4(\ln n)^2 \left(\sum_{y=0}^{s-1} \frac{(3c\ln\ln n)^y}{y!}\right)^h \leqslant$$

(assuming $0 < c < 1$)

$$\leqslant nc \left(\frac{(\ln\ln n)^2}{\ln n}\right)^{\frac{3\ln n}{\ln\ln n}} \left(\frac{n}{n-h}\right)^h 4(\ln n)^2 \left(e^{3c\ln\ln n}\right)^h \leqslant$$

$$\leqslant n \cdot c \cdot e^{-3\ln n + O(\ln n \ln\ln\ln n/\ln\ln n)}\, e^{O((\ln n)^2/n)} n^{O(\ln\ln n/\ln n)} e^{3c\ln n} = c \cdot n^{(3c-2)(1+o(1))}; \tag{36}$$

$$\frac{n}{h} \sum_{l=s+1}^{n} \frac{l+1}{l!} \left(\frac{h\Delta}{n\, r^{n-1}}\right)^{l+1} \leqslant \frac{2\Delta}{r^{n-1}} \sum_{l=s+1}^{n} \frac{1}{(l-1)!} \left(\frac{h\Delta}{n\, r^{n-1}}\right)^{l} \leqslant$$

(since $\Delta/r^{n-1} < cn$ and $(s+j)! > s!j!$)

$$\leqslant 2c\, n \frac{1}{s!} \left(\frac{h\Delta}{n\, r^{n-1}}\right)^{s+1} \sum_{l=0}^{n-s-1} \frac{1}{l!} \left(\frac{h\Delta}{n\, r^{n-1}}\right)^{l} \leqslant$$

(since $h\Delta/n\, r^{n-1} \leqslant 3c\ln\ln n$ and $s! > (s/e)^s$)

$$\leqslant \frac{2c\, ns}{e} \left(\frac{3ec\ln\ln n}{s}\right)^{s+1} \sum_{l=0}^{n-s-1} \frac{(3c\ln\ln n)^l}{l!} \leqslant \frac{2c\, ns}{e} \left(\frac{3ec\ln\ln n}{s}\right)^{s+1} e^{3c\ln\ln n} =$$

(since $s = \lfloor \ln n \rfloor$)

$$= c \cdot n^{1+O(\ln\ln n/\ln n)} e^{-(\ln\ln n)(\ln n)(1+o(1))} e^{O(\ln\ln n)} = c \cdot n^{-\ln\ln n(1+o(1))}. \tag{37}$$

The estimates (36) and (37) imply the following bound for the desired sum in the left-hand side of (33):

$$\sum_{\mathcal{J}\in\Psi_{\mathcal{Q}}} \mathsf{Pr}\,(\mathcal{J}) \leqslant |E(\mathcal{Q})| \Bigg((\Delta+1)r^{1-n}(1-t)^{n-h+1}(nt)^h + n \sum_{y_1,\dots,y_h=0}^{s-1} \left(\frac{h\Delta}{r^{n-1}n}\right)^{\sum_{i=1}^h y_i} \times$$
$$\times \left(\frac{\Delta}{r^{n-1}n}\right)^{h+1} \left(\frac{n}{n-h}\right)^h \frac{1+hs}{\prod_{i=1}^h (y_i!)} + \frac{n}{h}\sum_{l=s+1}^{n} \frac{(l+1)}{l!}\left(\frac{h\Delta}{n\,r^{n-1}}\right)^{l+1}\Bigg) \leqslant$$
$$\leqslant (n+1)(\Delta+1)r^{1-n}(1-t)^{n-h+1}(nt)^h + c\cdot n^{(3c-3)(1+o(1))} + c\cdot n^{1-\ln\ln n(1+o(1))}.$$

There exists an absolute constant $c\in(0,1)$ such that for any $n\geqslant 9$,

$$c\cdot n^{(3c-3)(1+o(1))} + c\cdot n^{1-\ln\ln n(1+o(1))} < 1/4.$$

After such choice of c we can take t very close to 1 satisfying

$$(n+1)(\Delta+1)r^{1-n}(1-t)^{n-h+1}(nt)^h + c\cdot n^{(3c-3)(1+o(1))} + c\cdot n^{1-\ln\ln n(1+o(1))} < 1/4.$$

Let us make final conclusions. By a special choice of the parameters we establish that the required inequality (28) holds for any $\mathcal{Q}\in\Psi(t)$. This relation implies that the Local Lemma can be applied to the system of the events $\Psi(t)$. The Local Lemma states that

$$\mathsf{Pr}\left(\bigcap_{\mathcal{Q}\in\Psi(t)} \overline{\mathcal{Q}}\right) > 0.$$

So, by (27) we showed that the probability of the event $\mathcal{F}(t)$ is strictly less than 1, and, consequently, hypergraph H is r-colorable. Theorem 4 is proved.

4 Proof of Theorem 3

Suppose that $N\leqslant c\cdot r^{n-1}\frac{(\ln\ln n)^2}{\ln n}$. We have to establish r-colorability of the hypergraph of arithmetic progressions $H_n(N)=([N],E_n(N))$. For this purpose, we shall apply the same random coloring $\zeta(t)$ from the proof of Theorem 4 to $H_n(N)$ and show that it is a proper r-coloring with positive probability. However, Theorem 4 requires the condition on girth (it should be greater than 5), so we have to make additional analysis for $H_n(N)$.

Before starting the analysis let us make some useful observations concerning arithmetic progressions.

Observation 1. *For any two integers x,y, there at most m^2 arithmetic progressions of length m containing both x and y.*

Observation 2. *For any arithmetic progression of length n, there are at most n^2m^2 arithmetic progressions of length m having at least two common vertices with it.*

For any arithmetic progression $A \in E_n(N)$, $A = \{a, a+d, \ldots, a+d(n-1)\}$, let us introduce a larger progression $l(A)$ of length $2n$ as follows:

$$l(A) = \left\{a - d\left\lfloor \frac{n}{2} \right\rfloor, \ldots, a, a+d, \ldots, a+d(n-1) + d\left\lceil \frac{n}{2} \right\rceil\right\}.$$

Observation 3. *If A and B are arithmetic progressions of length n such that $|A \cap B| \geqslant n/2$, then $B \subset l(A)$.*

The choice of the parameters of the probabilistic construction will be almost the same, in comparison with Theorem 4 we shall not take the time parameter t very close to 1:

$$h = \left\lfloor \frac{3 \ln n}{\ln \ln n} \right\rfloor, \quad s = \lfloor \ln n \rfloor, \quad t = \frac{2(\ln n)^2}{n}. \tag{38}$$

If t is small then for every vertex v, the probability that it can be recolored during the process is at most t, because X_v should be less than t. All these events are independent, so this fact helps to show (without constructing huge configurations) that the probability that an edge is monochromatic in the final coloring is sufficiently small.

Now we are ready to analyze bad events implied by short cycles in $H_n(N)$.

4.1 Short cycles in the hypergraph of arithmetic progressions

Let $(A_1, \ldots, A_j)$ be an ordered set of edges of $H_n(N)$, $j \geqslant 2$. It is said to form a *δ-j-cycle* if, for $j > 2$,

- $|A_i \cap A_{i+1}| = 1$ for $i = 1, \ldots, j-1$;
- $|A_i \cap A_k| = 0$ for $|k - i| > 1$, except the pair $(1, j)$;
- $1 \leqslant |A_1 \cap A_j| < n/2$ or $|A_1 \cap A_j| = 0$ and $|A_1 \cap l(A_j)| \geqslant 1$.

If $1 \leqslant |A_1 \cap A_j| < n/2$ then we have a normal cycle in a hypergraph. If $j = 2$ then we assume that $|l(A_1) \cap l(A_2)| \geqslant 2$ and $|A_1 \cap A_2| < n/2$.

For given δ-j-cycle $(A_1, \ldots, A_j)$, we define a bad event $\mathcal{L}_j(A_1, \ldots, A_j)$ as follows: every edge A_i is monochromatic or almost monochromatic in the initial coloring ξ. The probability of this event can be roughly estimated by choosing the vertices of dominating color in every edge:

$$\mathsf{Pr}\left(\mathcal{L}_j(A_1, \ldots, A_j)\right) \leqslant r^j \binom{n}{s}^j r^{js - |A_1 \cup \ldots \cup A_j|} \leqslant n^{js} r^{(j+2)s - (j-1/2)n}. \tag{39}$$

Every situation in the additional analysis of the bad event leads to the event $\mathcal{L}_j(A_1, \ldots, A_j)$ for some $j \leqslant 5$, so it would be sufficient to avoid only these events.

We will also need the estimate for the number of δ-j-cycles intersecting a fixed edge J. Suppose $A_i \cap J \neq \emptyset$ then

- A_i can be chosen in at most $\Delta = \Delta(H_n(N))$ ways;

- its number in the cycle — in at most j ways;
- the remained edges except A_j, if $i < j$ or A_1 for $i = j$ — in at most Δ^{j-2} ways;
- for $j > 2$, the $l(A_j)$ (or A_1) should intersect A_1 and A_{j-1} (or A_2 and $l(A_j)$), so it can be chosen (due to **Observation 1**) in at most $(2n)^4$ ways;
- for $j = 2$, we have $|l(A_1) \cap l(A_2)| \geqslant 2$, hence, **Observation 2** says that here we also have at most $(2n)^4$ variants.

Finally, we obtain that the total number of δ-j-cycles intersecting a fixed edge does not exceed

$$j(\Delta + 1)\Delta^{j-2}(2n)^4 \leqslant 32j\, n^4 \Delta^{j-1}. \tag{40}$$

4.2 Additional analysis of the first bad event

Recall that in the first bad event $\mathcal{A}(A, \Phi, \overrightarrow{y}, \Lambda)$ we have a "tree" with "trunk" A, "branches" $\Phi = (B_1, \dots, B_h)$ and "leaves" $\Lambda = (C_{1,1}, \dots, C_{1,y_1}, C_{2,1}, \dots, C_{h,1}, \dots, C_{h,y_h})$, where $C_{i,1}, \dots, C_{i,y_i}$ are the "leaves" of "branch" B_i. This event implies that A is monochromatic in the initial coloring ξ, B_i is almost monochromatic for any i, leaf–edges $C_{i,j}$ are also monochromatic in ξ. In Theorem 4 A, Φ and Λ form a real hypertree due to the condition on girth. So, for $H_n(N)$, we have to analyze the situations when the configuration contains short cycles.

2-cycles. Suppose $|A \cap B_i| \geqslant 2$ for some $i = 1, \dots, h$. It this case the event $\mathcal{A}(A, \Phi, \overrightarrow{y}, \Lambda)$ implies the event $\mathcal{L}_2(A, B_i)$. Note that if $|A \cap B_i| > s$ then the probability of the considered situation event is equal to zero, otherwise $|A \cap B_i| < n/2$. The case when $|B_i \cap C_{i,j}| \geqslant 2$ also implies the bad event $\mathcal{L}_2(C_{i,j}, B_i)$ since all leaf–edges are monochromatic in the random coloring ξ.

3-cycles. Now we can assume that $|A \cap B_i| = 1$ for any i, and $|B_i \cap C_{i,j}| = 1$ for any i, j. Suppose that $|B_i \cap B_k| > 0$ for some $i \neq k$. If $|B_i \cap B_k| \geqslant n/2$ then $|A \cap l(B_i)| \geqslant 2$ since A have one common vertex with both B_i and B_k and these vertices are distinct. Such situation implies the event $\mathcal{L}_2(A, B_i)$.

In the case $1 \leqslant |B_i \cap B_k| < n/2$ three edges A, B_i, B_k form a δ-3-cycle and we have got the event $\mathcal{L}_3(A, B_i, B_k)$.

The cases when $|C_{i,j} \cap A| > 0$ for some i, j, or $|C_{i,j} \cap C_{i,k}| > 0$ for some i and $j \neq k$ are analyzed in the same way and lead to the bad events of the type $\mathcal{L}_2$ or $\mathcal{L}_3$.

4-cycles. It remains to consider the situations when the leaf–edges from one branch intersect with another branch or leaf–edges from it. Suppose $|B_i \cap C_{k,j}| > 0$ for some $i \neq k$. If $|B_i \cap C_{k,j}| \geqslant n/2$ then $|l(C_{k,j}) \cap A| \geqslant 1$ and $|C_{k,j} \cap A| = 0$. Once again, we obtain the event $\mathcal{L}_3(A, B_k, C_{k,j})$ for δ-3-cycle $(A, B_k, C_{k,j})$.

Suppose now that $1 \leqslant |B_i \cap C_{k,j}| < n/2$. In this case we have δ-4-cycle $(B_i, A, B_k, C_{k,j})$ and appropriate event $\mathcal{L}_4$.

5-cycles. It remains to deal with 5-cycles assuming that there is no cycles of smaller size. Suppose that $|C_{i,j} \cap C_{k,f}| > 0$ for some $i \neq k$. If $|C_{i,j} \cap C_{k,f}| \geqslant n/2$ then $|l(C_{i,j}) \cap B_k| \geqslant 1$ and we get a δ-4-cycle $(B_k, A, B_i, C_{i,j})$ with the event $\mathcal{L}_4(B_k, A, B_i, C_{i,j})$.

Finally, if $1 \leqslant |C_{i,j} \cap C_{k,f}| < n/2$ then there is a δ-5-cycle $(C_{i,j}, B_i, A, B_k, C_{k,f})$ and the event $\mathcal{L}_5(C_{i,j}, B_i, A, B_k, C_{k,f})$ holds.

We finished the additional analysis of the first bad event $\mathcal{A}(A, \Phi, \overrightarrow{y}, \Lambda)$. The second one, $\mathcal{B}(A,t)$, depends only on one edge, so there is no need to deal with short cycles here. Now we proceed to the third bad event.

4.3 Additional analysis of the third bad event

The third bad event in the proof of Theorem 4 is $\mathcal{G}(A,t)$. Its analysis in paragraph 3.5 shows that for hypergraphs with large girth, it is covered by events $\mathcal{C}(A, l, \Phi)$ (see (23)), where A and $\Phi = (B_1, \ldots, B_l)$ form a hypertree with "trunk" A and "branches" $B_1, \ldots, B_l$. For the hypergraph of arithmetic progressions, this set of edges do not necessarily form a real hypertree, so we have to make an additional analysis. In comparison with the consideration of the first event here we shall use the time parameter t, this makes the probability of recoloring small for any fixed vertex. This is useful because we don't know the colors of the vertices of the edge A in the initial coloring.

2-cycles. Suppose that $|A \cap l(B_i)| \geqslant 2$ for some $i = 1, \ldots, l$ (e.g. $|A \cap B_i| \geqslant 2$). The event $\mathcal{C}(A, l, \Phi)$ implies that B_i is monochromatic in the initial coloring ξ and the event $\mathcal{G}(A,t)$ implies that A is monochromatic at the time t of another color. Thus, the following event $\mathcal{L}_6(A, B_i, t)$ holds:

- B_i is monochromatic in ξ of some color b;
- there is a color $a \neq b$ such that, for every $v \in A$, either $\xi_v = a$ or $\xi \neq a, \eta_v = a$ and $X_v < t$.

The probability of $\mathcal{L}_6(A, B_i)$ can be easily calculated:

$$\mathsf{Pr}\left(\mathcal{L}_6(A, B_i, t)\right) = r(r-1)r^{-n}\left(\frac{t}{r-1}\right)^{|A\cap B_i|}\left(\frac{1}{r}+\frac{t}{r}\right)^{n-|A\cap B_i|}.$$

If $|A \cap B_i| \leqslant n/2$ then

$$\left(\frac{1}{r}+\frac{t}{r}\right)^{n-|A\cap B_i|} \leqslant r^{-n/2}(1+t)^{n/2} \leqslant r^{-n/2}e^{tn/2}.$$

Otherwise $(t/(r-1))^{|A\cap B_i|} \leqslant (t/(r-1))^{n/2}$. Hence, in general case we obtain the following upper bound for the probability of $\mathcal{L}_6(A, B_i, t)$:

$$\mathsf{Pr}\left(\mathcal{L}_6(A, B_i, t)\right) \leqslant r^{2-n} \max\left\{(t/(r-1))^{n/2}, r^{-n/2}e^{tn/2}\right\}. \tag{41}$$

3-cycles. Assume that $|B_i \cap A| = 1$ and $|B_i \cap B_{i'} \cap A| = 0$ for any $i \neq i' \in \{1, \ldots, l\}$, but $|B_j \cap B_k| > 0$ for some $j \neq k$. If $|B_j \cap B_k| \geqslant n/2$ then $|A \cap l(B_i)| \geqslant 2$ and this situation is covered by the event $\mathcal{L}_6(A, B_i, t)$, which was considered before.

If $1 \leqslant |B_j \cap B_k| < n/2$ then (A, B_j, B_k) forms a δ-3-cycle and the following event $\mathcal{L}_7(A, B_j, B_k, t)$ holds:

- B_i and B_k are monochromatic in ξ of some color b;
- there is a color $a \neq b$ such that, for every $v \in A$, either $\xi_v = a$ or $\xi \neq a, \eta_v = a$ and $X_v < t$.

The probability of this event can be easily calculated:

$$\mathsf{Pr}\left(\mathcal{L}_7(A, B_j, B_k, t)\right) = r(r-1)\left(\frac{1}{r} + \frac{t}{r}\right)^{n-2}\left(\frac{t}{r-1}\right)^2 r^{-|B_j \cup B_k|} \leqslant r^{4-5n/2} e^{tn} t^2. \tag{42}$$

4.4 Completion of the proof

In comparison with Theorem 4 for hypergraph of arithmetic progressions $H_n(N)$, the set of bad events $\Psi(t)$ is larger since we add to it the events of the type $\mathcal{L}_i$, $i = 2, \ldots, 7$, to it. For any such new event $\mathcal{Q}$, the domain $D(\mathcal{Q})$ and the edge-set $E(\mathcal{Q})$ are defined similarly as it was done in paragraph 3.6. All we have to do is to check the condition (28) required for the application of the Local Lemma. By analogy with (29), for any $\mathcal{Q} \in \Psi(t)$, we have

$$\sum_{\mathcal{J} \in \Psi_{\mathcal{Q}}} \mathsf{Pr}\left(\mathcal{J}\right) \leqslant \sum_{U \in E(\mathcal{Q})} \left[\sum_{A \in E_n(N):\, U \cap A \neq \emptyset} \mathsf{Pr}\left(\mathcal{B}(A, t)\right) + \sum_{\substack{(A, \Phi, \overrightarrow{y}, \Lambda) \in \text{ CONF1:} \\ U \cap \left(A \cup \bigcup\limits_{B \in \Phi} B \cup \bigcup\limits_{C \in \Lambda} C\right) \neq \emptyset}} \mathsf{Pr}\left(\mathcal{A}(A, \Phi, \overrightarrow{y}, \Lambda)\right) + \right.$$

$$+ \sum_{\substack{(A, l, \Phi) \in \text{ CONF2:} \\ U \cap \left(A \cup \bigcup\limits_{B \in \Phi} B\right) \neq \emptyset}} \mathsf{Pr}\left(\mathcal{C}(A, l, \Phi)\right) + \sum_{j=2}^{5} \sum_{\substack{\delta - j - cycle\ (A_1, \ldots, A_j): \\ U \cap (A_1 \cup \ldots \cup A_j) \neq \emptyset}} \mathsf{Pr}\left(\mathcal{L}_j(A_1, \ldots, A_j)\right) +$$

$$\left. + \sum_{\substack{(A, B):\, |A \cap l(B)| \geqslant 2, \\ U \cap (A \cup B) \neq \emptyset}} \mathsf{Pr}\left(\mathcal{L}_6(A, B, t)\right) + \sum_{\substack{\delta - 3 - cycle\ (A, B, C): \\ U \cap (A \cup B \cup C) \neq \emptyset}} \mathsf{Pr}\left(\mathcal{L}_7(A, B, C, t)\right) \right]. \tag{43}$$

Before starting the analysis of the sums in the right-hand side of (43), let us make a preliminary observation. Let $\Delta = \Delta(H_n(N))$ denote the maximum edge degree of the hypergraph $H_n(N)$. Since any integer from $[N]$ is contained in at most $n(N-1)/(n-1)$ arithmetic progressions of length n from $E_n(N)$, we get

$$\Delta \leqslant \left(n \frac{N-1}{n-1} - 1\right) n \leqslant Nn \, \frac{n}{n-1} \leqslant \frac{9}{8} c\, r^{n-1} \frac{n (\ln \ln n)^2}{\ln n}. \tag{44}$$

Here we use the restriction on N and the condition $n \geqslant 9$.

1. The first sum in (43) is estimated by the same way as in (22) (the bound is correct since the choice of the parameter t implies the required inequality $nt > 2$):

$$\sum_{A\in E\colon U\cap A\neq\emptyset} \mathsf{Pr}\left(\mathcal{B}(A,t)\right) \leqslant (\Delta+1)r^{1-n}(1-t)^{n-h+1}(nt)^h \leqslant |\text{ using (44) and (38)}| \leqslant$$

$$\leqslant \frac{9}{4}\,cn\,e^{-t(n-h+1)}(2(\ln n)^2)^h = cn\,e^{-2(\ln n)^2(1+o(1))}e^{2h\ln\ln n(1+o(1))} = c\,e^{-2(\ln n)^2(1+o(1))}. \quad (45)$$

2. The second and the third sums were analyzed in paragraph 3.7, so the bounds (36) and (37) hold with c replaced by $(9/8)c$.

3. In the fourth sum the number of summands does not exceed the bound (40). Thus, by using estimate (39), we obtain

$$\sum_{j=2}^{5} \sum_{\substack{\delta-j-cycle\ (A_1,\ldots,A_j)\colon\\ U\cap(A_1\cup\ldots\cup A_j)\neq\emptyset}} \mathsf{Pr}\left(\mathcal{L}_j(A_1,\ldots,A_j)\right) \leqslant \sum_{j=2}^{5} 32j\,n^4\Delta^{j-1}n^{js}r^{(j+2)s-(j-1/2)n} \leqslant$$

(since $j\leqslant 5$, $s\leqslant \ln n$ and $\Delta \leqslant (9c/8)nr^{n-1}$)

$$\leqslant \sum_{j=2}^{5} 160\,(9c/8)^{j-1}n^{5\ln n+4}r^{7\ln n-1/2n} \leqslant c\,e^{5(\ln n)^2(1+o(1))}r^{7\ln n-1/2n} \leqslant$$

$$\leqslant |\text{ since } r\geqslant 2| \leqslant c\,e^{5(\ln n)^2(1+o(1))}2^{-n/2(1+o(1))} = c\,2^{-n/2(1+o(1))}. \quad (46)$$

4. In the fifth sum we have to calculate the number of pairs (A,B) intersecting a fixed progression U. The intersecting edge can be chosen by at most $2(\Delta+1)$ ways, but the second edge — in at most $(2n)^4$ ways since $|A\cap l(B)|\geqslant 2$. Hence, the estimate (41) implies that

$$\sum_{\substack{(A,B)\colon |A\cap l(B)|\geqslant 2,\\ U\cap(A\cup B)\neq\emptyset}} \mathsf{Pr}\left(\mathcal{L}_6(A,B,t)\right) \leqslant 2(\Delta+1)(2n)^4r^{2-n}\max\left\{(t/(r-1))^{n/2}, r^{-n/2}e^{tn/2}\right\}$$

(since $t=2(\ln n)^2/n$ and $\Delta\leqslant (9c/8)n\,r^{n-1}$)

$$\leqslant 64\cdot(9c/8)\,n^5\,r\cdot\max\left\{\left(\frac{2(\ln n)^2}{n(r-1)}\right)^{n/2}, r^{-n/2}e^{(\ln n)^2}\right\} \leqslant$$

$$\leqslant 72\,c\,n^5\,r\cdot\max\left\{\left(\frac{4(\ln n)^2}{nr}\right)^{n/2}, r^{-n/2}e^{(\ln n)^2}\right\} \leqslant$$

(it is easy to see that the second value is always maximal)

$$\leqslant 72\,c\,n^5\,r^{1-n/2}e^{(\ln n)^2} \leqslant 72\,c\,n^5\,2^{1-n/2}e^{(\ln n)^2} = c\,2^{-n/2(1+o(1))}. \quad (47)$$

5. Finally, for the last sum we know the estimate for the number of intersecting δ-3-cycles (40) and the the bound for the probability of the event (42). Thus,

$$\sum_{\substack{\delta-3-cycle\ (A,B,C):\\ U\cap(A\cup B\cup C)\neq\emptyset}} \mathsf{Pr}\left(\mathcal{L}_7(A,B,C)\right) \leqslant 96\Delta^2 n^4\, r^{4-5n/2} e^{tn} t^2 \leqslant$$

$$\leqslant |\text{ using (44)) and (38)}| \leqslant \frac{96\cdot 9^2}{8^2} c^2 n^6\, r^{2-n/2}\left(\frac{2\ln n}{n}\right)^2 = c^2 n^{4+o(1)} r^{2-n/2} \leqslant$$
$$\leqslant c^2 n^{4+o(1)} 2^{2-n/2} \leqslant c^2\cdot 2^{-n/2(1+o(1))}. \tag{48}$$

Let us complete the proof. For the application of the Local Lemma it is sufficient to show that the right-hand side of (43) does not exceed $1/4$. Since for any $\mathcal{Q}\in\Psi(t)$ $|E(\mathcal{Q})|\leqslant n+1$ (see (34)), by using the obtained estimates (45)–(48) we have

$$\sum_{\mathcal{J}\in\Psi_{\mathcal{Q}}} \mathsf{Pr}\left(\mathcal{J}\right) \leqslant (n+1)\left(c\,e^{-2(\ln n)^2(1+o(1))} + (9/8)c\cdot n^{(27c/8-2)(1+o(1))} + (9/8)c\cdot n^{-\ln\ln n(1+o(1))} + \right.$$
$$\left. + c\cdot 2^{-n/2(1+o(1))} + c\cdot 2^{-n/2(1+o(1))} + c^2\cdot 2^{-n/2(1+o(1))}\right).$$

It is easy to see that there exists $c\in(0,1)$ such that the given above function of n is strictly less than $1/4$ for all $n\geqslant 9$. Theorem 3 is proved.

5 Proof of Theorem 2

In the previous Section we used a reduction argument to establish r-colorability of the hypergraph of arithmetic progressions. For simple hypergraph H, this reduction is almost the same. There is no 2-cycles in H, so the bad events of the type $\mathcal{L}_1$, $\mathcal{L}_3$, $\mathcal{L}_6$ and $\mathcal{L}_4$ (for HYP2 case) could be omitted. The analysis of the remained bad events in Section 4 used only the following properties

- the maximum edge degree of the hypergraph is $O\left(r^{n-1}\frac{n(\ln\ln n)^2}{\ln n}\right)$ (condition (3) for H),
- the size of edge intersections is at most $n/2$ (at most 1 for a simple hypergraph),
- the maximum codegree is at most n^2 (equals 1 for a simple hypergraph).

All these properties hold for the hypergraph H under consideration, so the same proof argument provides its r-colorability. Theorem 2 is proved.

References

[1] N. Alon, J. H. Spencer, *The Probabilistic Method*, 3rd Edition, Wiley, 2008.

[2] J. Beck, "On 3-chromatic hypergraphs", *Discrete Mathematics*, **24**:2 (1978), 127–137.

[3] E. Berlekamp, "A construction for partitions which avoid long arithmetic progressions", *Canadian Mathematical Bulletin*, **11** (1968), 409–414.

[4] D.D. Cherkashin, J. Kozik, "A note on random greedy coloring of uniform hypergraphs", arXiv:1310.1368.

[5] P. Erdős, L. Lovász, "Problems and results on 3-chromatic hypergraphs and some related questions", *Infinite and Finite Sets*, Colloquia Mathematica Societatis Janos Bolyai, **10**, Amsterdam: North Holland, 1973, 609–627.

[6] P. Erdős, R. Radó, "Combinatorial theorems on classifications of subsets of given set", *Proceedings of London Mathematical Society*, **2** (1952), 417–439.

[7] W.T. Gowers, "A new proof of Szemerédi theorem", *Geometric and Functional Analysis*, **11** (2001), 465–588.

[8] R.L. Graham, B.L. Rotshild, J.H. Spencer, *Ramsey theory*, 2nd edition, Wiley, New York, 1990.

[9] B. Green, T. Tao, "New bounds for Szemerédi's theorem, II: a new bound for $r_4(N)$", *Analytic Number Theory: essays in honour of Klaus Roth*, eds. W. Chen et al, Cambridge University Press, 2009, 180–204.

[10] A.V. Kostochka, M. Kubmhat, "Coloring uniform hypergraphs with few edges", *Random Structures and Algorithms*, **35**:3 (2009), 348–368.

[11] A.V. Kostochka, M. Kumbhat, V. Rödl, "Coloring uniform hypergraphs with small edge degrees", *Fete of Combinatorics and Computer Science*, Bolyai Society Mathematical Studies, **20**, Springer, 2010, 213–238.

[12] A.V. Kostochka, V. Rödl, "Constructions od sparse uniform hypergraphs with high chromatic number", *Random Structures and Algorithms*, **36**:1 (2010), 46–56.

[13] L. Moser, "On a theorem of van der Waerden", *Canadian Mathematical Bulletin*, **3** (1960), 23–25.

[14] K. O'Bryant, "Sets of integers that do not contain long arithmetic progressions", *Electronic Journal of Combinatorics*, **18** (2011), research paper №59.

[15] J. Radhakrishnan, A. Srinivasan, "Improved bounds and algorithms for hypergraph two-coloring", *Random Structures and Algorithms*, **16**:1 (2000), 4–32.

[16] T. Sanders, "On Roth's theorem on progressions", *Annals of Mathematics*, **174**:1 (2011), 619–636.

[17] W. M. Schmidt, "Two combinatorial theorems on arithmetic progressions", *Duke Mathematical Journal*, **29**:1 (1962), 129–140.

[18] D.A. Shabanov, "On r-chromatic hypergraphs", *Discrete Mathematics*, **312**:2 (2012), 441–458.

[19] D.A. Shabanov, "Random coloring method in the combinatorial problem of Erdős and Lovász", *Random Structures and Algorithms*, **40**:2 (2012), c. 227–253.

[20] D. A. Shabanov, "Van der Waerden's function and colourings of hypergraphs", *Izvestiya: Mathematics*, **75**:5 (2011), 1063–1091.

[21] J. H. Spencer, "Coloring n-sets red and blue", *J. Combinatorial Theory, Series A*, **30**:1 (1981), 112–113.

[22] Z. Szabó, "An application of Lovasz Local Lemma - a new lower bound for the van der Waerden number", *Random Structures and Algorithms*, **1**:3 (1990), 343–360.

[23] B.L. van der Waerden, "Beweis einer Baudetschen Vermutung", *Nieuw Archief voor Wiskunde*, **15** (1927), 212–216.