Simultaneous approximation of polynomials

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Abstract. Let $\mathcal{P}_d$ denote the family of all polynomials of degree at most d in one variable x, with real coefficients. A sequence of positive numbers $x_1 \leq x_2 \leq \ldots$ is called *$\mathcal{P}_d$-controlling* if there exist $y_1, y_2, \ldots \in \mathbb{R}$ such that for every polynomial $p \in \mathcal{P}_d$ there exists an index i with $|p(x_i) - y_i| \leq 1$. We settle a problem of Makai and Pach (1983) by showing that $x_1 \leq x_2 \leq \ldots$ is $\mathcal{P}_d$-controlling if and only if $\sum_{i=1}^{\infty} \frac{1}{x_i^d}$ is divergent. The proof is based on a statement about covering the Euclidean space with translates of slabs, which is related to Tarski's plank problem.

1 Introduction

Let $\mathcal{F}$ be a class of real functions $\mathbb{R} \to \mathbb{R}$. We say that a sequence of positive numbers $x_1, x_2, \ldots$ is $\mathcal{F}$-*controlling* if there exist reals $y_1, y_2, \ldots$ with the property that for every $f \in \mathcal{F}$, one can find an i with

$$|f(x_i) - y_i| \leq 1.$$

In other words, a sequence $x_1, x_2, \ldots$ is $\mathcal{F}$-controlling if we can find $y_1, y_2, \ldots \in \mathbb{R}$ such that the points $p_1 = (x_1, y_1), p_2 = (x_2, y_2), \ldots \in \mathbb{R}^2$ simultaneously approximate all functions in $\mathcal{F}$, in the sense that the graph of every member $f \in \mathcal{F}$ gets (vertically) not farther than 1 to at least one point p_i. In this paper, we address the following question raised in [11]. Given a class of functions $\mathcal{F}$, how *sparse* an $\mathcal{F}$-controlling sequence can be? A similar question, motivated by a problem of László Fejes Tóth [5], was studied in [4].

Let $\mathcal{P}_d$ denote the class of polynomials $\mathbb{R} \to \mathbb{R}$ of degree at most d. It was shown by Makai and Pach [11] that if a sequence of positive numbers $x_1 \leq x_2 \leq \ldots$ is $\mathcal{P}_d$-controlling, then the infinite series $\frac{1}{x_1^d} + \frac{1}{x_2^d} + \ldots$ is *divergent*. They conjectured that this condition is also sufficient for a sequence $x_1 \leq x_2 \leq \ldots$ to be $\mathcal{P}_d$-controlling (see Conjecture 3.2.B in [11]). The aim of this note is to prove this statement.

Theorem 1 *Let d be a positive integer and $x_1 \leq x_2 \leq \ldots$ be a monotone increasing infinite sequence of positive numbers. The sequence $x_1, x_2, \ldots$ is* $\mathcal{P}_d$-controlling *if and only if $\frac{1}{x_1^d} + \frac{1}{x_2^d} + \frac{1}{x_3^d} \ldots = \infty$.*

We also generalize this result to other finitely generated function classes. Given $d+1$ real functions, $f_0, f_1, \ldots, f_d : \mathbb{R}_+ \to \mathbb{R}_+$, let $\mathcal{L} = \mathcal{L}(f_0, \ldots, f_d)$ denote the set of all functions that can be obtained as *linear combinations* of them with real coefficients. That is,

$$\mathcal{L} = \{a_0 f_0 + \ldots + a_d f_d : a_0, \ldots, a_d \in \mathbb{R}\}.$$

Here $\mathbb{R}_+$ stands for the set of positive reals.

Theorem 2 *Let $d \geq 1$ be an integer, $x_0 > 0$, $\epsilon > 0$, and let $f_0, f_1, \ldots, f_d : \mathbb{R}_+ \to \mathbb{R}_+$ be real functions that are monotone increasing for $x \geq x_0$ and bounded over every bounded subinterval of $\mathbb{R}_+$. Assume that the functions $F_j(x) = f_j(x)/(f_d(x))^{1-\epsilon}$ $(j = 0, \ldots, d-1)$ are monotone decreasing for $x \geq x_0$ and tend to 0 as $x \to \infty$.*

An increasing sequence of positive numbers $x_1 \leq x_2 \leq \ldots$ is $\mathcal{L}(f_0, \ldots, f_d)$-controlling if and only if $\sum_{i=1}^{\infty} \frac{1}{f_d(x_i)} = \infty$.

Obviously, the functions $f_i(x) = x^i$ $(i = 0, 1, \ldots, d))$ meet the above requirements, so that Theorem 2 implies Theorem 1.

For the proof of Theorem 1, we will rephrase the question as a covering problem for slabs. A *slab* (sometimes called *plank* or *strip*) is the set of points S lying between two parallel hyperplanes in $\mathbb{R}^d$. The distance w between these two hyperplanes is called the *width* of the slab. We can write S as

$$S = \{\mathbf{x} \in \mathbb{R}^d : b - \frac{w}{2} \leq \langle \mathbf{v}, \mathbf{x} \rangle \leq b + \frac{w}{2}\},$$

for some unit vector $\mathbf{v}$ and real number b. We say that a sequence of slabs $S_1, S_2, \ldots$ permits a *translative covering* of a subset $\mathbb{R}^d$ if there are suitable translates S'_i of S_i $(i = 1, 2, \ldots)$ such that $\cup_{i=1}^{\infty} S'_i = \mathbb{R}^d$.

As it was shown in [11], Theorem 1 (and, in fact, Theorem 2, too) would easily follow from

Conjecture 1 ([11], [3]) *Let d be a positive integer. A sequence of slabs in $\mathbb{R}^d$ with widths $w_1, w_2, \ldots$ permits a translative covering of $\mathbb{R}^d$ if and only if $\sum_{i=1}^{\infty} w_i = \infty$.*

The fact that this condition is *necessary* follows, for example, from Tarski's result [12] which states that the total width of any system of slabs the union of which covers a disk of unit diameter is at least 1. Tarski's "plank problem," whether this statement remains true in higher dimensions, remained open for almost twenty years. In 1950, Bang [1,**?**] answered this question in the affirmative. For $d = 2$, Conjecture 1 was proved by Makai and Pach [11] and, according to [6], independently, by Erdős and Straus (unpublished). (See [7,8] for some refinements.) For $d \geq 3$, the problem is open. Groemer [6] proved that any sequence of slabs in $\mathbb{R}^d$ with widths $w_1, w_2, \ldots$ satisfying

$$\sum_{i=1}^{\infty} w_i^{\frac{d+1}{2}} = \infty$$

permits a translative covering of $\mathbb{R}^d$. Recently, the authors of the present note [9] have come close to settling Conjecture 1 by replacing Groemer's *sufficient* condition with the weaker assumption

$$\limsup_{n \to \infty} \frac{w_1 + w_2 + \ldots + w_n}{\log(1/w_n)} > 0.$$

In particular, any sequence of slabs of widths $1, \frac{1}{2}, \frac{1}{3}, \ldots$ permits a translative covering of space.

To establish Theorem 1, it is enough to verify Conjecture 1 for special sequences of slabs, whose normal vectors lie on a moment curve. We will do precisely this in Section 2, by exploring the natural ordering of these vectors. In Section 3, we generalize our arguments to establish Theorem 2. The last section contains a few concluding remarks.

2 Proof of Theorem 1

We only have to prove the "if" part of the theorem.

Let $x_1 \leq x_2 \leq \ldots$ be a monotone increasing sequence of positive numbers with $\sum_i \frac{1}{x_i^d} = \infty$. We have to find a sequence of reals $y_1, y_2, \ldots$ such that for any polynomial $p(x) = \sum_{j=0}^{d} a_j x^j$ with real coefficients a_j, there exists a positive integer i with $|p(x_i) - y_i| \leq 1$. Write $p(x)$ in the form $p(x) = \langle \mathbf{x}, \mathbf{a} \rangle$, where $\mathbf{x} = (1, x, \ldots, x^d)$, $\mathbf{a} = (a_0, a_1, \ldots, a_d) \in \mathbb{R}^{d+1}$, and $\langle . \rangle$ stands for the scalar product. Using this notation, we have $\mathbf{x}_i = (1, x_i, \ldots, x_i^d)$ and the inequality $|p(x_i) - y_i| \leq 1$ can be rewritten as

$$y_i - 1 \leq \langle \mathbf{x}_i, \mathbf{a} \rangle \leq y_i + 1.$$

For a fixed i, the locus of points $\mathbf{a} \in \mathbb{R}^{d+1}$ satisfying this double inequality is a slab $S_i \subset \mathbb{R}^{d+1}$ of width $w_i = \frac{2}{\|\mathbf{x}_i\|} = \frac{2}{(\sum_{j=0}^{d} x_i^{2j})^{1/2}}$, with normal vector $\mathbf{x}_i$. See Fig. 1. Since the condition $\sum_{i=1}^{\infty} \frac{1}{x_i^d} = \infty$ is equivalent

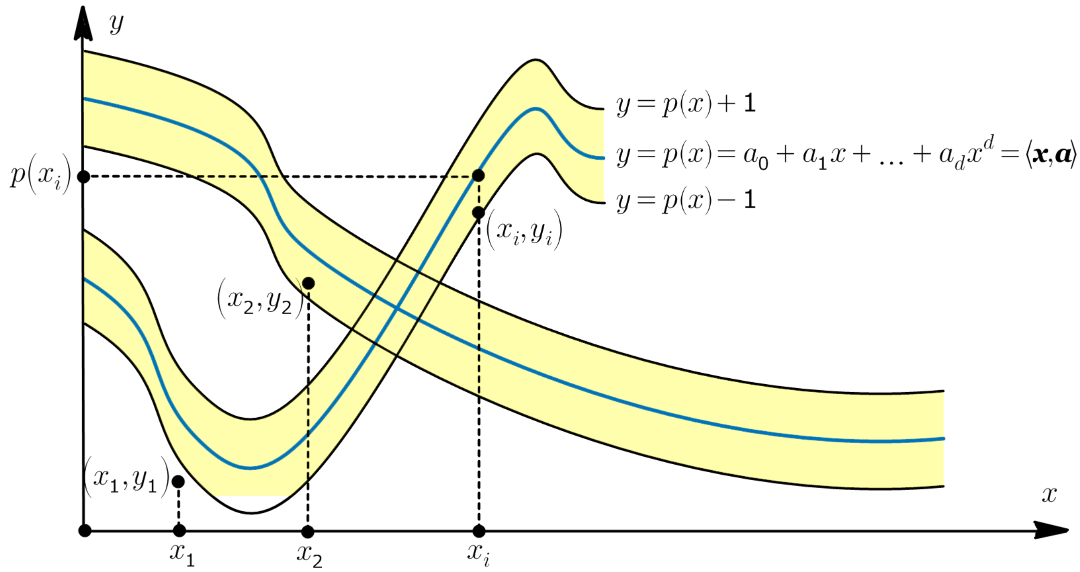

Fig. 1. Controlling polynomials of degree at most d.

to $\sum_{i=0}^{\infty} \frac{2}{||\mathbf{x}_i||} = \infty$, the sequence $x_1, x_2, \ldots$ is $\mathcal{P}_d$-controlling if and only if the sequence of slabs $S_1, S_2, \ldots$ permits a translative covering of $\mathbb{R}^{d+1}$.

If $x_i \le 3$ for infinitely many (and, hence, for all) positive integers i, then for the widths of the corresponding slabs S_i we have $w_i > \frac{1}{3^d}$. Thus, these slabs permit a translative covering of $\mathbb{R}^{d+1}$, because each of them can be translated to cover any ball of diameter $\frac{1}{3^d}$.

Therefore, we can assume that $x_i > 3$ for all $i \ge m$. In fact, we can assume without loss of generality that $x_i > 3$ for all $i \ge 1$, otherwise we simply discard the first $m-1$ members of the sequence, and prove the theorem for the resulting sequence $x_m \le x_{m+1} \le \ldots$.

We are going to exploit the fact that the normal vectors $\mathbf{x}_i = (1, x_i, \ldots, x_i^d)$ of the slabs S_i lie on the moment curve $(1, x, x^2, \ldots, x^d)$. First, we need an auxiliary lemma.

Lemma 1 *Let d be a positive integer, let $3 \le x_1 \le x_2 \le \ldots$ be a finite or infinite sequence of reals, and let $\mathbf{x}_i = (1, x_i, x_i^2, \ldots, x_i^d)$ for every i. Then there exist $d+1$ linearly independent vectors $\mathbf{u}_1, \ldots, \mathbf{u}_{d+1} \in \mathbb{R}^{d+1}$ such that for every i $(i = 1, 2, \ldots)$ and j $(j = 1, 2, \ldots, d+1)$, we have*

$$(i) \qquad \frac{\langle \mathbf{x}_{i+1}, \mathbf{u}_1 \rangle}{\langle \mathbf{x}_i, \mathbf{u}_1 \rangle} \le \frac{\langle \mathbf{x}_{i+1}, \mathbf{u}_j \rangle}{\langle \mathbf{x}_i, \mathbf{u}_j \rangle},$$

$$(ii) \qquad \langle \mathbf{x}_i, \mathbf{u}_j \rangle \ge \frac{1}{3} \|\mathbf{x}_i\| \|\mathbf{u}_j\|.$$

Proof. Take the standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_{d+1}$ in $\mathbb{R}^{d+1}$, i.e., let $\mathbf{e}_i$ denote the all-zero vector with a single 1 at the i-th position. Set $\mathbf{u}_j := \mathbf{e}_{d+1-j} + \mathbf{e}_{d+1}$ for $j = 1, \ldots, d$ and $\mathbf{u}_{d+1} := \mathbf{e}_{d+1}$.

Condition (i) trivially holds for $j = 1$ and very easy to check for $j = d+1$. For $j = 2, \ldots, d$, it reduces to

$$\frac{x_{i+1}^{d-1} + x_{i+1}^d}{x_i^{d-1} + x_i^d} \le \frac{x_{i+1}^{d-j} + x_{i+1}^d}{x_i^{d-j} + x_i^d},$$

which is equivalent to

$$(x_{i+1}^{d-1}+x_{i+1}^{d})(x_i^{d-j}+x_i^d) \le (x_{i+1}^{d-j}+x_{i+1}^{d})(x_i^{d-1}+x_i^d).$$

The last inequality can be rewritten as

$$x_{i+1}^{d-j}x_i^{d-j}(x_{i+1}-x_i)(\sum_{k=0}^{j-1}x_{i+1}^k x_i^{j-1-k}+\sum_{k=0}^{j-2}x_{i+1}^k x_i^{j-2-k}-x_{i+1}^{j-1}x_i^{j-1}) \le 0,$$

or, dividing both sides by $x_{i+1}^{d-j}x_i^{d-j}(x_{i+1}-x_i)$, as

$$\sum_{k=0}^{j-1}x_{i+1}^k x_i^{j-1-k}+\sum_{k=0}^{j-2}x_{i+1}^k x_i^{j-2-k}-x_{i+1}^{j-1}x_i^{j-1} \le 0.$$

Using the fact $x_{i+1} \ge x_i$, and bounding from above each sum by its largest term multiplied by the number of terms, we obtain that the left-hand side of the last inequality is at most

$$jx_{i+1}^{j-1}+(j-1)x_{i+1}^{j-2}-x_{i+1}^{j-1}x_i^{j-1} < x_{i+1}^{j-1}(2j-1-x_i^{j-1}).$$

As $x_i \ge 3$, the right-hand side of this inequality is always negative and (i) holds.

It remains to verify condition (ii). Taking into account that $x_i \ge 3$, we have

$$\langle \mathbf{x}_i, \mathbf{u}_{d+1}\rangle = x_i^d \ge \frac{1}{2}\|\mathbf{x}_i\| = \frac{1}{2}\|\mathbf{x}_i\|\|\mathbf{u}_{d+1}\|.$$

On the other hand, for $j=1,\dots,d$, we obtain

$$\langle \mathbf{x}_i, \mathbf{u}_j\rangle = x_i^{d-j}+x_i^d \ge \frac{1}{2}\|\mathbf{x}_i\| \ge \frac{1}{3}\|\mathbf{x}_i\|\|\mathbf{u}_j\|.$$

This completes the proof of Lemma 1. □

In order to establish Theorem 1, it is enough to prove that there is a constant c depending on d such that any system of slabs S_i $(i=1,\dots,n)$ in $\mathbb{R}^{d+1}$ whose normal vectors are $(1, x_i, \dots, x_i^d)$ for some $3 \le x_1 \le x_2 \le \dots \le x_n$ and whose total width is at least c, permits a translative covering of a ball of unit diameter. This is an immediate corollary of Lemma 1 and the following assertion.

Lemma 2 *For every positive integer d, for any system of $d+1$ linearly independent vectors $\mathbf{u}_1,\dots,\mathbf{u}_{d+1}$ in $\mathbb{R}^{d+1}$, and for any $\gamma > 0$, there is a constant c with the following property.*

Given any system of slabs S_i $(i=1,\dots,n)$ in $\mathbb{R}^{d+1}$, whose normal vectors $\mathbf{x}_i$ satisfy the conditions

$$(i) \qquad \frac{\langle \mathbf{x}_{i+1}, \mathbf{u}_1\rangle}{\langle \mathbf{x}_i, \mathbf{u}_1\rangle} \le \frac{\langle \mathbf{x}_{i+1}, \mathbf{u}_j\rangle}{\langle \mathbf{x}_i, \mathbf{u}_j\rangle},$$

$$(ii) \qquad \langle \mathbf{x}_i, \mathbf{u}_j\rangle \ge \gamma\|\mathbf{x}_i\|\|\mathbf{u}_j\|$$

for every i and j, and whose total width $\sum_{i=1}^n w_i$ is at least c, the slabs S_i permit a translative covering of a $(d+1)$-dimensional ball of unit diameter.

Proof. Instead of covering a ball of unit diameter, it will be more convenient to cover the simplex Δ with one vertex in the origin $\mathbf{0}$ and the others at the points (vectors) $\mathbf{u}_j$ $(j=1,\dots,d+1)$. By properly scaling these vectors, if necessary, we can assume that Δ contains a ball of unit diameter.

We place the slabs one by one. See Fig. 2. We place S_1', a translate of S_1, so that one of its boundary hyperplanes passes through $\mathbf{0}$ and the other one cuts a simplex Δ_1 out of the cone Γ of all linear combinations

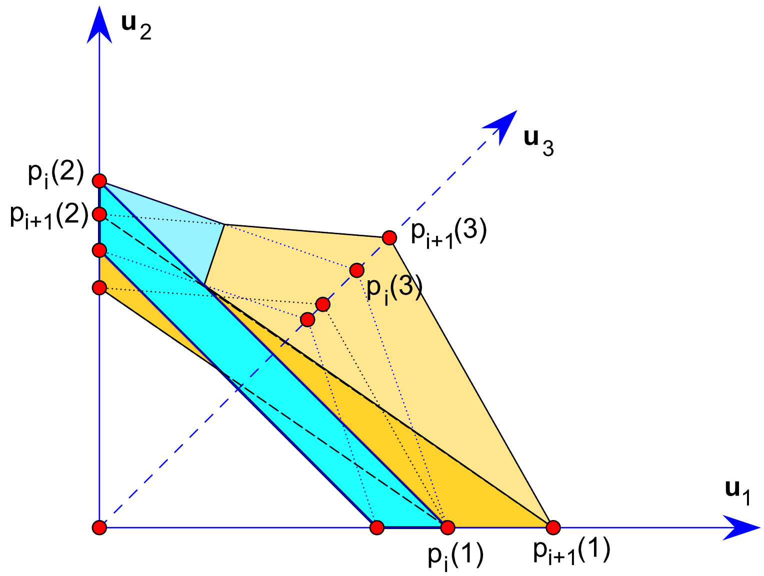

Fig. 2. We place the slabs one by one.

of the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_{d+1}$ with positive coefficients. According to assumption (ii), we have $\langle \mathbf{x}_1, \mathbf{u}_j \rangle > 0$ for every j. Therefore, S'_1 does not separate Γ into two cones: $S'_1 \cap \Gamma$ is indeed a simplex Δ_1.

Suppose that we have already placed $S'_1, \ldots, S'_i$, the translates of $S_1, \ldots, S_i$, so that their union covers a simplex Δ_i with one vertex at the origin, and the others along the $d+1$ half-lines that span the cone Γ. We also assume that the facet of Δ_i opposite to the origin is a boundary hyperplane of S'_i. Let $\mathbf{p}_i(j)$ denote the vertex of Δ_i that belongs to the open half-line parallel to $\mathbf{u}_j$ emanating from 0 ($j = 1, \ldots, d+1$).

Next, we place a translate S'_{i+1} of S_{i+1} so that one of its boundary hyperplanes, denoted by π, passes through $\mathbf{p}_i(1)$, and the other one, π', cuts the half-line parallel to $\mathbf{u}_1$ at a point $\mathbf{p}_{i+1}(1)$ with $\|\mathbf{p}_{i+1}(1)\| > \|\mathbf{p}_i(1)\|$. That is, $\mathbf{p}_{i+1}(1)$ is further away from the origin than $\mathbf{p}_i(1)$ is. Let $\mathbf{p}_{i+1}(2), \ldots, \mathbf{p}_{i+1}(d+1)$ denote the intersection points of π' with the half-lines parallel to $\mathbf{u}_2, \ldots, \mathbf{u}_{d+1}$, respectively, and let Δ_{i+1} be the simplex induced by the vertices $\mathbf{0}, \mathbf{p}_{i+1}(1), \ldots, \mathbf{p}_{i+1}(d+1)$.

We have to verify that Δ_{i+1} is entirely covered by the slabs $S'_1, \ldots, S'_{i+1}$. By the induction hypothesis, Δ_i was covered by the slabs $S'_1, \ldots, S'_i$. Thus, it is sufficient to check that the hyperplane π intersects every edge $\mathbf{0}\mathbf{p}_i(j)$ of Δ_i, for $j = 1, \ldots, d+1$. Let $\alpha_j \mathbf{u}_j$ be the intersection point of π with the half-line parallel to $\mathbf{u}_j$, and let $\mathbf{p}_i(j) = \beta_j \mathbf{u}_j$. We have to prove that $\alpha_j \leq \beta_j$.

By definition, we have $\langle \mathbf{x}_{i+1}, \mathbf{p}_i(1) - \alpha_j \mathbf{u}_j \rangle = 0$ and $\langle \mathbf{x}_i, \mathbf{p}_i(1) - \beta_j \mathbf{u}_j \rangle = 0$. From here, we get

$$\frac{\alpha_j}{\beta_j} = \frac{\langle \mathbf{x}_{i+1}, \mathbf{p}_i(1) \rangle}{\langle \mathbf{x}_{i+1}, \mathbf{u}_j \rangle} \Big/ \frac{\langle \mathbf{x}_i, \mathbf{p}_i(1) \rangle}{\langle \mathbf{x}_i, \mathbf{u}_j \rangle} = \frac{\langle \mathbf{x}_{i+1}, \mathbf{p}_i(1) \rangle}{\langle \mathbf{x}_i, \mathbf{p}_i(1) \rangle} \Big/ \frac{\langle \mathbf{x}_{i+1}, \mathbf{u}_j \rangle}{\langle \mathbf{x}_i, \mathbf{u}_j \rangle} = \frac{\langle \mathbf{x}_{i+1}, \mathbf{u}_1 \rangle}{\langle \mathbf{x}_i, \mathbf{u}_1 \rangle} \Big/ \frac{\langle \mathbf{x}_{i+1}, \mathbf{u}_j \rangle}{\langle \mathbf{x}_i, \mathbf{u}_j \rangle}.$$

In view of assumption (i) of the lemma, the right-hand side of the above chain of equations is at most 1, as required.

Observe that during the whole procedure the uncovered part of the cone Γ always remains convex and, hence, connected. In the nth step, $\cup_{i=1}^{n} S'_i \supset \Delta_n$. By the construction, $\mathbf{p}_i(1)$ lies at least w_i farther away from the origin along the half-line parallel to $\mathbf{u}_1$ than $\mathbf{p}_{i-1}(1)$ does. Thus, we have

$$\|\mathbf{p_n}(1)\| \geq \sum_{i=1}^{n} w_i \geq c.$$

Using the fact that $\langle \mathbf{x}_n, \mathbf{p}_n(j) - \mathbf{p}_n(1)\rangle = 0$ for every $j \geq 2$, and taking into account assumption (ii), we obtain

$$\|\mathbf{p}_n(j)\| \geq \frac{\langle \mathbf{x}_n, \mathbf{p}_n(j)\rangle}{\|\mathbf{x}_n\|} = \frac{\langle \mathbf{x}_n, \mathbf{p}_n(1)\rangle}{\|\mathbf{x}_n\|} \geq \gamma\|\mathbf{p_n}(1)\| \geq \gamma c.$$

Thus, if c is sufficiently large, we have $\|\mathbf{p}_n(j)\| \geq \|\mathbf{u}_j\|$. This means that Δ_n contains the simplex Δ defined in the first paragraph of this proof. Hence, it also contains a ball of unit diameter, as required. □

3 Proof of Theorem 2

In this section, we extend the technique used in the proof of Theorem 1 to establish Theorem 2.

As in the proof Theorem 1, we can write any function $l = \sum_{k=0}^{d} a_k f_k \in \mathcal{L}(f_0, \ldots, f_d)$ as $l(x) = \langle \mathbf{x}, \mathbf{a}\rangle$, where $\mathbf{x} = (f_0(x), f_1(x), \ldots, f_d(x))$ and $\mathbf{a} = (a_0, a_1, \ldots, a_d) \in \mathbb{R}^{d+1}$. As before, we only have to prove the "if" part of the theorem, which is equivalent to the fact that the slabs $S_i \subset \mathbb{R}^{d+1}$ with normal vector $\mathbf{x}_i = (f_0(x_i), \ldots, f_d(x_i))$ and width

$$w_i = \frac{2}{\|\mathbf{x}_i\|} = \frac{2}{(\sum_{k=0}^{d} f_k^2(x_i))^{1/2}} \geq \frac{2}{\sqrt{d} f_d(x_i)},$$

for $i = 1, 2, \ldots$, permit a translative covering of $\mathbb{R}^{d+1}$. Again, it is enough to consider the case when $\lim_{i\to\infty} x_i = \infty$, otherwise each slab S_i contains a ball of diameter at least

$$\frac{2}{\sqrt{d} f_d(\lim_{i\to\infty} x_i)} > 0.$$

We follow the scheme of the proof of Theorem 1. According to Lemma 2, it is enough to show that there exist $d+1$ linearly independent vectors $\mathbf{u_1}, \ldots, \mathbf{u_{d+1}}$ that satisfy conditions (i) and (ii) with $\mathbf{x}_i = (f_0(x_i), \ldots, f_d(x_i))$ and with a suitable constant $\gamma > 0$. We can assume without loss of generality that x_1, and hence all x_is, are so large that they satisfy $x_1 \geq x_0$ and the inequalities

$$\frac{f_j(x)}{f_d(x)} \leq \frac{f_j(x_1)}{f_d(x_1)} \leq \frac{1}{\sqrt{d}}, \tag{1}$$

for every $x \geq x_1$ and $j = 0, \ldots, d-1$. To see this, observe that $f_j(x)/f_d(x) = F_j(x)/f_d^{\epsilon}(x)$ is monotone decreasing in x, because F_j is monotone decreasing, while f_d is monotone increasing.

Let $\mathbf{e}_1, \ldots, \mathbf{e}_{d+1}$ be the standard basis in $\mathbb{R}^{d+1}$. For $1 \leq j \leq d+1$, set

$$\mathbf{u}_j := \sum_{k=1}^{d+1} \mathbf{e}_k - \frac{1}{2}\mathbf{e}_{d+2-j}.$$

In other words, all coordinates of $\mathbf{u}_j$ are 1, with the exception of the $(d+2-j)$-th coordinate, which is $\frac{1}{2}$.

By definition, we have $\langle \mathbf{x}_i, \mathbf{u}_j\rangle \geq \frac{1}{2} f_d(x_i)$ and $\|\mathbf{u}_j\| < \sqrt{d+1}$. It follows from (1) that $\frac{f_j(x_i)}{f_d(x_i)} \leq \frac{1}{\sqrt{d}}$ for $j \neq d$, so that

$$\|\mathbf{x}_i\| \leq \left(\sum_{k=0}^{d} f_k^2(x_i)\right)^{1/2} \leq \sqrt{2} f_d(x_i).$$

Hence, for every i and j,

$$\langle \mathbf{x}_i, \mathbf{u}_j\rangle \geq \frac{1}{2} f_d(x_i) \geq \frac{1}{2\sqrt{2}}\|\mathbf{x}_i\| \geq \frac{1}{2\sqrt{2(d+1)}}\|\mathbf{x}_i\|\|\mathbf{u}_j\|.$$

Therefore, condition (ii) in Lemma 2 is satisfied with $\gamma = \frac{1}{2\sqrt{2(d+1)}}$.

It remains to verify condition (i). For the rest of the argument, fix j ($1 \le j \le d+1$). We have to show that for every i ($i = 1, 2, \ldots$), the inequality

$$\frac{\langle \mathbf{x}_{i+1}, \mathbf{u}_1 \rangle}{\langle \mathbf{x}_i, \mathbf{u}_1 \rangle} \le \frac{\langle \mathbf{x}_{i+1}, \mathbf{u}_j \rangle}{\langle \mathbf{x}_i, \mathbf{u}_j \rangle}$$

holds. For $j = 1$, the statement is trivial. Therefore, we may suppose that $j > 1$. Next, we want to get rid of $f_d(x)$ in the left hand side, keeping both the numerator and denominator positive. The above inequality is equivalent to the following:

$$\frac{\langle \mathbf{x}_{i+1}, \mathbf{u}_1 \rangle - \frac{1}{2}\langle \mathbf{x}_{i+1}, \mathbf{u}_j \rangle}{\langle \mathbf{x}_i, \mathbf{u}_1 \rangle - \frac{1}{2}\langle \mathbf{x}_i, \mathbf{u}_j \rangle} \le \frac{\langle \mathbf{x}_{i+1}, \mathbf{u}_j \rangle}{\langle \mathbf{x}_i, \mathbf{u}_j \rangle}.$$

Using the notation

$$\phi(x) = \frac{1}{2}\sum_{k=0}^{d-1} f_k(x) + \frac{1}{4} f_{d+1-j}(x), \qquad \psi(x) = \sum_{k=0}^{d-1} f_k(x) - \frac{1}{2} f_{d+1-j}(x),$$

the above inequality may be rewritten as

$$\frac{\phi(x_{i+1})}{\phi(x_i)} \le \frac{f_d(x_{i+1}) + \psi(x_{i+1})}{f_d(x_i) + \psi(x_i)}. \tag{2}$$

Before checking that (2) is true, let us summarize the properties of the functions ϕ and ψ we need:

1. $\phi(x_{i+1})/\phi(x_i) \le f_d^{1-\epsilon}(x_{i+1})/f_d^{1-\epsilon}(x_i)$ for the constant $\epsilon > 0$ from Theorem 2,
2. $\psi(x_{i+1}) \le c f_d^{1-\epsilon}(x_{i+1})$ for a constant $c > 0$, and
3. $\psi(x_{i+1}) \ge \psi(x_i)$.

By the monotonicity of F_k, we have $f_k(x_{i+1})/f_k(x_i) \le f_d^{1-\epsilon}(x_{i+1})/f_d^{1-\epsilon}(x_i)$, for $k = 0, \ldots, d-1$. Now property 1 follows from the fact that, if $a_0, \ldots, a_{d-1}, b_0, \ldots, b_{d-1}, t$ are positive numbers satisfying $a_0/b_0 \le t, \ldots, a_{d-1}/b_{d-1} \le t$, then $(a_0 + \ldots + a_{d-1})/(b_0 + \ldots + b_{d-1}) \le t$. Using that $\lim_{x\to\infty} F_k(x) = 0$ for $k = 0, \ldots, d-1$, we get property 2. Property 3 is a direct consequence of our assumption that each f_k ($k = 0, 1, \ldots$) is monotone increasing for $x \ge x_0$.

We have to verify (2). In view of property 1, it is sufficient to show

$$\frac{f_d^{1-\epsilon}(x_{i+1})}{f_d^{1-\epsilon}(x_i)} \le \frac{f_d(x_{i+1}) + \psi(x_{i+1})}{f_d(x_i) + \psi(x_i)},$$

which is equivalent to

$$\psi(x_i) f_d^{1-\epsilon}(x_{i+1}) - \psi(x_{i+1}) f_d^{1-\epsilon}(x_i) \le f_d(x_i) f_d^{1-\epsilon}(x_{i+1}) \left(\left(\frac{f_d(x_{i+1})}{f_d(x_i)} \right)^{\epsilon} - 1 \right),$$

or, in a slightly different form,

$$\psi(x_i) f_d^{1-\epsilon}(x_{i+1}) - \psi(x_{i+1}) f_d^{1-\epsilon}(x_i) \le f_d(x_i) f_d^{1-\epsilon}(x_{i+1}) \left(\left(1 + \frac{f_d^{1-\epsilon}(x_{i+1}) - f_d^{1-\epsilon}(x_i)}{f_d^{1-\epsilon}(x_i)} \right)^{\frac{\epsilon}{1-\epsilon}} - 1 \right).$$

Replacing the left-hand side by a larger quantity (taking property 3 into account) and the right-hand side by a smaller one (applying the inequality $(1+x)^\alpha \ge 1 + \alpha x$, valid for all $\alpha, x \ge 0$), we obtain the stronger inequality

$$\psi(x_{i+1})(f_d^{1-\epsilon}(x_{i+1}) - f_d^{1-\epsilon}(x_i)) \le f_d(x_i) f_d^{1-\epsilon}(x_{i+1}) \left(\frac{\epsilon}{1-\epsilon} \frac{f_d^{1-\epsilon}(x_{i+1}) - f_d^{1-\epsilon}(x_i)}{f_d^{1-\epsilon}(x_i)} \right). \tag{3}$$

Thus, it is sufficient to prove (3). Rearranging the terms, we obtain

$$\psi(x_{i+1}) \le \frac{\epsilon}{1-\epsilon} f_d^{\epsilon}(x_i) f_d^{1-\epsilon}(x_{i+1}).$$

By property 2, we have $\psi(x_{i+1}) \le c f_d^{1-\epsilon}(x_{i+1})$, so that it is enough to check that

$$c f_d^{1-\epsilon}(x_{i+1}) \le \frac{\epsilon}{1-\epsilon} f_d^{\epsilon}(x_i) f_d^{1-\epsilon}(x_{i+1}),$$

that is, $c \le \frac{\epsilon}{1-\epsilon} f_d^{\epsilon}(x_i)$. As $f_d(x)$ is an increasing function for $x \ge x_0$, the last inequality is satisfied if we choose x_1 (and, hence, all other x_i) sufficiently large.

This completes the proof of (3), (2), and so the proof of Theorem 2. □

4 Concluding remarks

1. As was mentioned in the Introduction, Conjecture 1 is known to be true in the plane. Moreover, in [11] a stronger statement was proved: there exists a constant c such that every collection of strips with total width at least c permits a translative covering of a disk of diameter 1. In view of this, one can make the following even bolder conjecture.

Conjecture 2 *For any positive integer d, there exists a constant c depending on d such that every collection of slabs in $\mathbb{R}^d$ of total width at least c permits a translative covering of a unit diameter d-dimensional ball.*

Suppose Conjecture 1 is true for a positive integer d. Answering a question in [11], Imre Z. Ruzsa [10] proved that then, for the same value of d, Conjecture 2 also holds. Thus, the two conjectures are equivalent.

2. Given a class $\mathcal{F}$ of functions $\mathbb{R} \to \mathbb{R}$, we say that a sequence of positive numbers $x_1 \le x_2 \le \dots$ is *strongly $\mathcal{F}$-controlling* if there exist reals $y_1, y_2, \dots$ with the property that, for every $\varepsilon > 0$ and every $f \in \mathcal{F}$, one can find an i with

$$|f(x_i) - y_i| \le \varepsilon.$$

It is easy to see that the condition in Theorem 1 is sufficient to guarantee that the sequence $x_1, x_2, \dots$ is strongly $\mathcal{P}_d$-controlling. Theorem 2 can also be strengthened analogously.

3. The aim of this paper was to find necessary and sufficient conditions for a sequence of numbers to be $\mathcal{L}$-controlling, where $\mathcal{L} = \mathcal{L}(f_1, \dots, f_d)$ is the class of functions that can be obtained as linear combinations of $f_1, \dots, f_d$. We reduced this problem to a question about covering $\mathbb{R}^d$ with translates of certain slabs. However, the two problems are not necessarily equivalent. For example, we have noticed that the slabs obtained at this reduction had some special properties: apart from their widths, their normal vectors were also prescribed. This enabled us to cover $\mathbb{R}^d$ with their translates, even if we do not know whether such a covering exists for *every* system of slabs with the same widths.

Nevertheless, in a more complicated sense, the two problems are equivalent.

Theorem 3 *Given a positive integer d, and a sequence of positive numbers $x_1, x_2, \dots$, define a family $\mathcal{F} = \mathcal{F}(d, x_1, x_2, \dots)$ of d-tuples of functions $f_1, \dots, f_d : \mathbb{R} \to \mathbb{R}$ as*

$$\mathcal{F} = \{(f_1, \dots, f_d) : \sum_{j=1}^{d} f_j^2(x_i) = x_i^2 \text{ for all } i\}.$$

Then a sequence of slabs with widths $x_1, x_2, \dots$ permits a translative covering of $\mathbb{R}^d$ if and only if $x_1, x_2, \dots$ is $\mathcal{L}(f_1, \dots, f_d)$-controlling for every d-tuple $(f_1, \dots, f_d) \in \mathcal{F}$, where

$$\mathcal{L}(f_1, \dots, f_d) = \{a_1 f_1 + \dots + a_d f_d : a_1, \dots, a_d \in \mathbb{R}\}.$$

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References

1. Th. Bang, *On covering by parallel-strips*, Mat. Tidsskr. B. **1950** (1950), 49–53.
2. Th. Bang, *A solution of the "plank problem,"* Proc. Amer. Math. Soc. **2** (1951), 990–993.
3. P. Brass, W. Moser, and J. Pach, *Research Problems in Discrete Geometry*, Springer, Heidelberg, 2005.
4. P. Erdős and J. Pach, *On a problem of L. Fejes Tóth,* Discrete Math. **30** (1980), no. 2, 103–109.
5. L. Fejes Tóth, *Remarks on the dual of Tarski's plank problem* in Hungarian), Matematikai Lapok **25** (1974), 13–20.
6. H. Groemer, *On coverings of convex sets by translates of slabs,* Proc. Amer. Math. Soc. **82** (1981), no. 2, 261–266.
7. H. Groemer, *Covering and packing properties of bounded sequences of convex sets*, Mathematika **29** (1982), 18–31.
8. H. Groemer, *Some remarks on translative coverings of convex domains by strips*, Canad. Math. Bull. **27** (1984), no. 2, 233–237.
9. A. Kupavskii and J. Pach, *Translative covering of the space with slabs*, manuscript.
10. I. Z. Ruzsa, personal communication.
11. E. Makai Jr. and J. Pach, *Controlling function classes and covering Euclidean space*, Stud. Scient. Math. Hungarica **18** (1983), 435–459.
12. A. Tarski, *Uwagi o stopniu równoważności wielokątów* (in Polish), Parametr **2** (1932), 310–314.