A short proof for an extension of the Erdős-Ko-Rado Theorem

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Dedicated to the 80th birthday of Ron Graham

Abstract

A proof with almost no computation is given for the following inequality due to Pyber. If $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$ satisfy $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}, B \in \mathcal{B}$ and $n \geq 2k$ then $|\mathcal{A}||\mathcal{B}| \leq \binom{n-1}{k-1}^2$ holds.

Introduction

Let n, k be positive integers with $n \geq 2k$ and let $\binom{[n]}{k}$ denote the collection of all k-subsets of $[n] = \{1, \ldots, n\}$. Two families $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$ are said to be *cross-intersecting* if $A \cap B \neq \emptyset$ holds for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

The aim of this short note is to give a short, simple proof of the following theorem of Pyber.

Theorem 1 (Pyber [10])**.** *If $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$ are cross-intersecting and $n \geq 2k$, then one has*

$$|\mathcal{A}||\mathcal{B}| \leq \binom{n-1}{k-1}^2. \tag{1}$$

Let us mention that in the case $\mathcal{A} = \mathcal{B}$ (1) implies $|\mathcal{A}| \leq \binom{n-1}{k-1}$, which is the classical Erdős-Ko-Rado Theorem [2]. For various proofs of it let us refer to a paper of Ron and the first author [5]. One of them is due to Daykin [1], who observed that the Erdős-Ko-Rado Theorem can be deduced from the Kruskal-Katona Theorem ([8], [7]) on shadows. Hilton [6] observed that the same theorem can be used to investigate cross-intersecting families. To present his result we need to introduce the so-called *lexicographic order* $\prec$ on k-element sets.

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Definition 1. *$A, B \in \binom{[n]}{k}$, $A \prec B$ iff the minimal element of $A \setminus B$ is smaller than that of $B \setminus A$.*

Definition 2. *For $0 \le m \le \binom{n}{k}$ let $\mathcal{L}(m)$ denote the first m sets from $\binom{[n]}{k}$ in the lexicographic order.*

Lemma 2 (Hilton, [6])**.** *If $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$ are cross-intersecting then so are $\mathcal{L}(|\mathcal{A}|)$ and $\mathcal{L}(|\mathcal{B}|)$ as well.*

By Hilton's lemma, in order to prove (1), it is sufficient to consider the case when the set families are initial segments in the lexicographic order. That is what we shall do in the next section.

The proof of Theorem 1

By symmetry, we suppose $|\mathcal{A}| \le |\mathcal{B}|$. First note that if $|\mathcal{A}| \le \binom{n-2}{k-2}$, then

$$|\mathcal{A}||\mathcal{B}| \le \binom{n-2}{k-2}\binom{n}{k} = \binom{n-1}{k-1}^2 \frac{n}{k} \cdot \frac{k-1}{n-1}.$$

As $n(k-1) = nk - n < nk - k = (n-1)k$, (1) holds in this case with strict inequality.

From now on we assume that $\binom{n-2}{k-2} \le |\mathcal{A}| \le |\mathcal{B}|$. By Lemma 2 we suppose that $\mathcal{A} = \mathcal{L}(|\mathcal{A}|)$, $\mathcal{B} = \mathcal{L}(|\mathcal{B}|)$, i.e., both families are initial segments in the lexicographic order.

Note that the first $\binom{n-2}{k-2}$ sets in the lexicographic order are all the k-sets that contain 1 and 2. Since $\mathcal{A}, \mathcal{B}$ are cross-intersecting, we infer that all their members must contain either 1 or 2. We shall use this fact to prove:

Proposition 3. *We have*

$$|\mathcal{A}| + |\mathcal{B}| \le 2\binom{n-1}{k-1}. \tag{2}$$

Note that (2) implies (1) by the inequality between arithmetic and geometric means. One can even deduce that (1) is strict unless $|\mathcal{A}| = |\mathcal{B}| = \binom{n-1}{k-1}$ holds.

Proof of Proposition 3. If $|\mathcal{B}| \le \binom{n-1}{k-1}$ then (2) is obvious. Therefore, we assume $|\mathcal{B}| > \binom{n-1}{k-1}$. Note that the first $\binom{n-1}{k-1}$ members of $\binom{[n]}{k}$ are all the k-sets containing 1. Since $\mathcal{A}, \mathcal{B}$ are cross-intersecting, $1 \in A$ holds for all $A \in \mathcal{A}$. Let $\mathcal{B}'$ be the family of the remaining sets in $\mathcal{B}$, i.e.,

$$\mathcal{B}' = \{B \in \mathcal{B} : 1 \notin B\}.$$

Let $\mathcal{C} = \{C \in \binom{[n]}{k} : 1 \in C, C \notin \mathcal{A}\}$. To prove (2) we need to show that

$$|\mathcal{C}| \ge |\mathcal{B}'| \quad \text{holds.}$$

Recall that *all* k-sets containing both 1 and 2 are in $\mathcal{A}$ and therefore all members of $\mathcal{B}$ contain 1 or 2. We infer that $B \cap \{1, 2\} = \{2\}$ for all $B \in \mathcal{B}'$ and $C \cap \{1, 2\} = \{1\}$ for all $C \in \mathcal{C}$.

Let us now consider a bipartite graph $\mathcal{G} = (X_1, X_2, E)$, where $X_i := \left\{D_i \in \binom{[n]}{k} : D_i \cap \{1,2\} = \{i\}\right\}$ and two vertices D_1 and D_2 are connected by an edge if and only if $D_1 \cap D_2 = \emptyset$ holds.

Note that $\mathcal{G}$ is regular of degree $\binom{n-k-1}{k-1}$, $\mathcal{C} \subseteq X_1, \mathcal{B}' \subseteq X_2$ hold. Moreover, the cross-intersecting property implies that if D_1 and D_2 are connected for some $D_2 \in \mathcal{B}'$ then $D_1 \in \mathcal{C}$. In other words, the full neighborhood of $\mathcal{B}'$ in the regular bipartite graph $\mathcal{G}$ is contained in $\mathcal{C}$. This implies $|\mathcal{C}| \geq |\mathcal{B}'|$ and concludes the proof. □

Some remarks

Matsumoto and Tokushige [9] proved the following extension of (1).

Theorem 4 ([9])**.** *If n, k, l are positive integers with $n \geq 2k, n \geq 2l$ and $\mathcal{A} \subset \binom{[n]}{k}, \mathcal{B} \subset \binom{[n]}{l}$ are cross-intersecting then*

$$|\mathcal{A}||\mathcal{B}| \leq \binom{n-1}{k-1}\binom{n-1}{l-1} \quad \textit{holds.} \tag{3}$$

Their proof, as the proof of Pyber, involves some non-trivial, rather lengthy computation. It would be nice to have a shorter, more elegant argument. We succeeded in using some general results from a recent paper [4] to somewhat shorten the calculations leading to (3). However, it is still far from being elegant.

From our proof of (1) it follows that equality is achieved only if $|\mathcal{A}| = |\mathcal{B}| = \binom{n-1}{k-1}$. Using a result of Füredi and Griggs [5] concerning uniqueness in the Kruskal-Katona Theorem it follows that the only way to achieve equality in (1) is letting both $\mathcal{A}$ and $\mathcal{B}$ consist of all k-subsets containing a fixed element.

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