Covering the space by slabs

Andrey B. Kupavskii

EPFL, Lausanne and MIPT, Moscow

Joint work with Janos Pach

Introduction

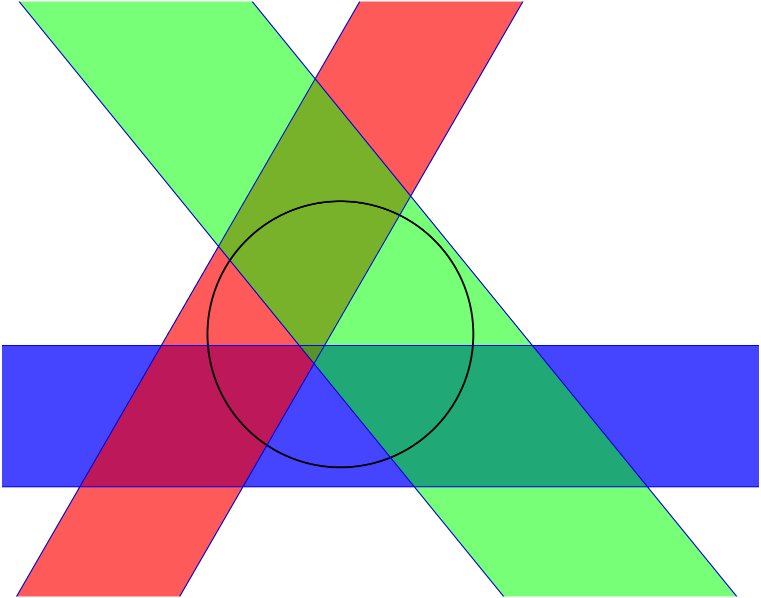
The set of points S lying between two parallel hyperplanes in $\mathbb{R}^d$ at distance w from each other is called a **slab** (a **strip** in $\mathbb{R}^2$) of **width** w.

The problem we tackle: Given a sequence of slabs, we aim to cover the whole plane or a unit ball with their translates.

Introduction

The set of points S lying between two parallel hyperplanes in $\mathbb{R}^d$ at distance w from each other is called a **slab** (a **strip** in $\mathbb{R}^2$) of **width** w.

The problem we tackle: Given a sequence of slabs, we aim to cover the whole plane or a unit ball with their translates.



Makai and Pach, Erdős and Straus, Groemer

There is a constant c such that any system of slabs in the plane with total width at least c permits a translative covering of a disk of diameter 1.

Corollary: any sequence of slabs with divergent total weight permits a translative covering of the plane.

What about higher dimensions?

Makai and Pach, Erdős and Straus, Groemer

There is a constant c such that any system of slabs in the plane with total width at least c permits a translative covering of a disk of diameter 1.

Corollary: any sequence of slabs with divergent total weight permits a translative covering of the plane.

What about higher dimensions?

Slabs

Makai and Pach, Erdős and Straus, Groemer

There is a constant c such that any system of slabs in the plane with total width at least c permits a translative covering of a disk of diameter 1.

Corollary: any sequence of slabs with divergent total weight permits a translative covering of the plane.

What about higher dimensions?

Conjecture (Makai-Pach)

Let d be a positive integer. A sequence of slabs in $\mathbb{R}^d$ with widths $w_1, w_2, \ldots$ permits a translative covering of $\mathbb{R}^d$ if and only if $\sum_{i=1}^{\infty} w_i = \infty$.

Groemer: It is true provided $\sum_{i=1}^{\infty} w_i^{\frac{d+1}{2}} = \infty$.

Theorem 1 (Kupavskii-Pach)

It is true if $w_1 \geqslant w_2 \geqslant \ldots$ is a monotone decreasing infinite sequence of positive numbers such that

$$\limsup_{n\to\infty} \frac{w_1 + w_2 + \ldots + w_n}{\log(1/w_n)} > 0.$$

Conjecture (Makai-Pach)

Let d be a positive integer. A sequence of slabs in $\mathbb{R}^d$ with widths $w_1, w_2, \ldots$ permits a translative covering of $\mathbb{R}^d$ if and only if $\sum_{i=1}^{\infty} w_i = \infty$.

Groemer: It is true provided $\sum_{i=1}^{\infty} w_i^{\frac{d+1}{2}} = \infty$.

Theorem 1 (Kupavskii-Pach)

It is true if $w_1 \geqslant w_2 \geqslant \ldots$ is a monotone decreasing infinite sequence of positive numbers such that

$$\limsup_{n \to \infty} \frac{w_1 + w_2 + \ldots + w_n}{\log(1/w_n)} > 0.$$

Makai-Pach conjecture

Conjecture (Makai-Pach)

Let d be a positive integer. A sequence of slabs in $\mathbb{R}^d$ with widths $w_1, w_2, \ldots$ permits a translative covering of $\mathbb{R}^d$ if and only if $\sum_{i=1}^{\infty} w_i = \infty$.

Groemer: It is true provided $\sum_{i=1}^{\infty} w_i^{\frac{d+1}{2}} = \infty$.

Theorem 1 (Kupavskii-Pach)

It is true if $w_1 \geqslant w_2 \geqslant \ldots$ is a monotone decreasing infinite sequence of positive numbers such that

$$\limsup_{n\to\infty} \frac{w_1 + w_2 + \ldots + w_n}{\log(1/w_n)} > 0.$$

Theorem (Tarski, 1932)

The total width of any system of strips that cover a disk in $\mathbb{R}^2$ of unit diameter is at least 1.

Bang, 1950: The same holds for balls in higher dimensions.

Intermission: Tarski's plank problem

Theorem (Tarski, 1932)

The total width of any system of strips that cover a disk in $\mathbb{R}^2$ of unit diameter is at least 1.

Bang, 1950: The same holds for balls in higher dimensions.

- Instead of covering the space, cover a unit ball with a finite portion of the sequence.
- Make the slabs twice thinner and cover a large fraction of the ball with "thin" slabs.
- The remaining part is so small that in each of its point's ϵ-neighborhood there is a point that is covered.
- Blow up the slabs. Now everything is covered.

Outline of the proof of Theorem 1

- Instead of covering the space, cover a unit ball with a finite portion of the sequence.
- Make the slabs twice thinner and cover a large fraction of the ball with “thin” slabs.
- The remaining part is so small that in each of its point’s ϵ-neighborhood there is a point that is covered.
- Blow up the slabs. Now everything is covered.

Outline of the proof of Theorem 1

- Instead of covering the space, cover a unit ball with a finite portion of the sequence.
- Make the slabs twice thinner and cover a large fraction of the ball with "thin" slabs.
- The remaining part is so small that in each of its point's ϵ-neighborhood there is a point that is covered.
- Blow up the slabs. Now everything is covered.

Outline of the proof of Theorem 1

- Instead of covering the space, cover a unit ball with a finite portion of the sequence.
- Make the slabs twice thinner and cover a large fraction of the ball with "thin" slabs.
- The remaining part is so small that in each of its point's ϵ-neighborhood there is a point that is covered.
- Blow up the slabs. Now everything is covered.

Let $\mathcal{F}$ be a class of real functions $\mathbb{R} \to \mathbb{R}$. We say that a sequence of positive numbers $x_1, x_2, \dots$ is *$\mathcal{F}$-controlling* if there exist reals $y_1, y_2, \dots$ with the property that for every $\ell \in \mathcal{F}$, one can find an i with

$$|f(x_i) - y_i| \leqslant 1.$$

Let $\mathcal{P}_d$ denote the class of polynomials $\mathbb{R} \to \mathbb{R}$ of degree at most d.

Controlling polynomials

Let $\mathcal{F}$ be a class of real functions $\mathbb{R} \to \mathbb{R}$. We say that a sequence of positive numbers $x_1, x_2, \ldots$ is *$\mathcal{F}$-controlling* if there exist reals $y_1, y_2, \ldots$ with the property that for every $\ell \in \mathcal{F}$, one can find an i with

$$|f(x_i) - y_i| \leqslant 1.$$

Let $\mathcal{P}_d$ denote the class of polynomials $\mathbb{R} \to \mathbb{R}$ of degree at most d.

Theorem 2 (Kupavskii-Pach)

Let d be a positive integer and $0 < x_1 \leqslant x_2 \leqslant \ldots$ be a monotone increasing infinite sequence of positive numbers. The sequence $x_1, x_2, \ldots$ is $\mathcal{P}_d$-*controlling* if and only if

$$\lim_{n\to\infty} (x_1^{-d} + x_2^{-d} + \ldots + x_n^{-d}) = \infty.$$

- Take a polynomial $p(x) = \sum_{j=0}^{d} a_j x^j$. $p(x) = \langle \mathbf{x}, \mathbf{a} \rangle$, where $\mathbf{x} = (1, x, \ldots, x^d)$, $\mathbf{a} = (a_0, a_1, \ldots, a_d) \in \mathbb{R}^{d+1}$, $\langle , \rangle$ stands for the scalar product.
- Using this notation, $|p(x_i) - y_i| \leqslant 1$ can be rewritten as

$$y_i - 1 \leqslant \langle \mathbf{x}_i, \mathbf{a} \rangle \leqslant y_i + 1.$$

- Assuming $|x_i| \geqslant 2$, the locus of points $\mathbf{a} \in \mathbb{R}^{d+1}$ satisfying this double inequality is a slab of width $w_i = \frac{2}{\|\mathbf{x}_i\|} \geqslant \frac{1}{|x_i|^d}$.

Reduction to a slab problem

- Take a polynomial $p(x) = \sum_{j=0}^{d} a_j x^j$. $p(x) = \langle \mathbf{x}, \mathbf{a} \rangle$, where $\mathbf{x} = (1, x, \ldots, x^d)$, $\mathbf{a} = (a_0, a_1, \ldots, a_d) \in \mathbb{R}^{d+1}$, $\langle , \rangle$ stands for the scalar product.
- Using this notation, $|p(x_i) - y_i| \leqslant 1$ can be rewritten as

$$y_i - 1 \leqslant \langle \mathbf{x}_i, \mathbf{a} \rangle \leqslant y_i + 1.$$

- Assuming $|x_i| \geqslant 2$, the locus of points $\mathbf{a} \in \mathbb{R}^{d+1}$ satisfying this double inequality is a slab of width $w_i = \frac{2}{\|\mathbf{x}_i\|} \geqslant \frac{1}{|x_i|^d}$.

- Take a polynomial $p(x) = \sum_{j=0}^{d} a_j x^j$. $p(x) = \langle \mathbf{x}, \mathbf{a} \rangle$, where $\mathbf{x} = (1, x, \dots, x^d)$, $\mathbf{a} = (a_0, a_1, \dots, a_d) \in \mathbb{R}^{d+1}$, $\langle , \rangle$ stands for the scalar product.
- Using this notation, $|p(x_i) - y_i| \leqslant 1$ can be rewritten as

$$y_i - 1 \leqslant \langle \mathbf{x}_i, \mathbf{a} \rangle \leqslant y_i + 1.$$

- Assuming $|x_i| \geqslant 2$, the locus of points $\mathbf{a} \in \mathbb{R}^{d+1}$ satisfying this double inequality is a slab of width $w_i = \frac{2}{\|\mathbf{x}_i\|} \geqslant \frac{1}{|x_i|^d}$.

Proof of Theorem 2.

- By pigeon-hole principle select the slabs that are almost parallel.
- Instead of covering a unit ball, we cover an appropriate simplex.
- The simplex is formed by a basis $\mathbf{u}_1, \ldots, \mathbf{u}_{d+1}$ in $\mathbb{R}^{d+1}$, that satisfy

$$(i) \qquad \frac{\langle \mathbf{x}_{i+1}, \mathbf{u}_1 \rangle}{\langle \mathbf{x}_i, \mathbf{u}_1 \rangle} \leqslant \frac{\langle \mathbf{x}_{i+1}, \mathbf{u}_j \rangle}{\langle \mathbf{x}_i, \mathbf{u}_j \rangle},$$

$$(ii) \qquad \langle \mathbf{x}_i, \mathbf{u}_j \rangle \geqslant 1/2 \|\mathbf{x}_i\| \|\mathbf{u}_j\|.$$

Proof of Theorem 2.

- By pigeon-hole principle select the slabs that are almost parallel.
- Instead of covering a unit ball, we cover an appropriate simplex.
- The simplex is formed by a basis $\mathbf{u}_1, \dots, \mathbf{u}_{d+1}$ in $\mathbb{R}^{d+1}$, that satisfy

$$(i) \qquad \frac{\langle \mathbf{x}_{i+1}, \mathbf{u}_1 \rangle}{\langle \mathbf{x}_i, \mathbf{u}_1 \rangle} \leqslant \frac{\langle \mathbf{x}_{i+1}, \mathbf{u}_j \rangle}{\langle \mathbf{x}_i, \mathbf{u}_j \rangle},$$

$$(ii) \qquad \langle \mathbf{x}_i, \mathbf{u}_j \rangle \geqslant 1/2 \|\mathbf{x}_i\| \|\mathbf{u}_j\|.$$

Proof of Theorem 2.

- By pigeon-hole principle select the slabs that are almost parallel.
- Instead of covering a unit ball, we cover an appropriate simplex.
- The simplex is formed by a basis $\mathbf{u}_1, \ldots, \mathbf{u}_{d+1}$ in $\mathbb{R}^{d+1}$, that satisfy

$$(i) \qquad \frac{\langle \mathbf{x}_{i+1}, \mathbf{u}_1 \rangle}{\langle \mathbf{x}_i, \mathbf{u}_1 \rangle} \leqslant \frac{\langle \mathbf{x}_{i+1}, \mathbf{u}_j \rangle}{\langle \mathbf{x}_i, \mathbf{u}_j \rangle},$$

$$(ii) \qquad \langle \mathbf{x}_i, \mathbf{u}_j \rangle \geqslant 1/2 \|\mathbf{x}_i\| \|\mathbf{u}_j\|.$$

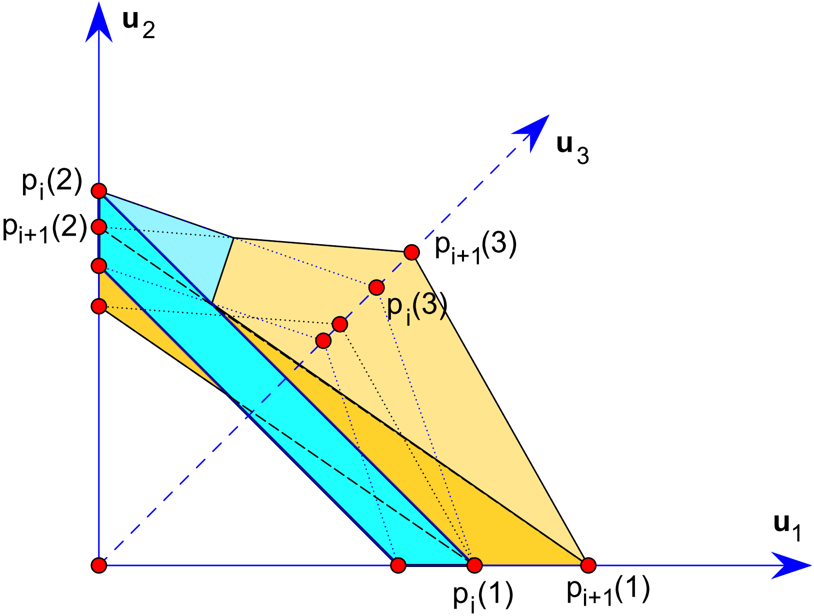
Proof of Theorem 2.

- Ensure the growth of the simplex by increasing one of its sides at each step. Make sure that the region covered stay convex.

- Choose a unit ball inside the simplex. It is possible since the simplex is not flat and one of the sides is large.

- Ensure the growth of the simplex by increasing one of its sides at each step. Make sure that the region covered stay convex.

- Choose a unit ball inside the simplex. It is possible since the simplex is not flat and one of the sides is large.


u2
u3
u1
$p_i(2)$
$p_{i+1}(2)$
$p_{i+1}(3)$
$p_i(3)$
$p_i(1)$
$p_{i+1}(1)$

Open problems

Makai-Pach

Is it true that a sequence of slabs in $\mathbb{R}^d$ with widths $w_1, w_2, \ldots$ permits a translative covering of $\mathbb{R}^d$ iff $\sum_{i=1}^{\infty} w_i = \infty$?

Conditions on controlling sequences for Lipschitz functions $\mathbb{R}^m \to \mathbb{R}^n$.

Makai-Pach

Is it true that a sequence of slabs in $\mathbb{R}^d$ with widths $w_1, w_2, \dots$ permits a translative covering of $\mathbb{R}^d$ iff $\sum_{i=1}^{\infty} w_i = \infty$?

Conditions on controlling sequences for Lipschitz functions $\mathbb{R}^m \to \mathbb{R}^n$.