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On Schur's Conjecture in $\mathbb{R}^4$

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Abstract—A diameter graph in $\mathbb{R}^d$ is a graph in which vertices are points of a finite subset of $\mathbb{R}^d$ and two vertices are joined by an edge if the distance between them is equal to the diameter of the vertex set. This paper is devoted to Schur's conjecture, which asserts that any diameter graph on n vertices in $\mathbb{R}^d$ contains at most n complete subgraphs of size d. It is known that Schur's conjecture is true in dimensions $d \leq 3$. We prove this conjecture for $d = 4$ and give a simple proof for $d = 3$.

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1. INTRODUCTION

1.1. History of the Question

This paper is concerned with a classical object of combinatorial geometry, diameter graphs [1]. A *diameter graph in* $\mathbb{R}^d$ is a graph in which vertices are points of a bounded subset of $\mathbb{R}^d$ and two vertices are joined by an edge if the distance between them is equal to the diameter of the vertex set. In what follows, without loss of generality, we consider only diameter graphs with vertex set of diameter 1.

The interest in diameter graphs arose in the 1930s, largely thanks to Borsuk's well-known problem (see [2]–[4]). In 1933, the renowned topologist Borsuk [5] proved that the d-sphere cannot be partitioned into d pieces of smaller diameter and posed the question: Is it true that any set of diameter 1 can be partitioned into $d+1$ pieces of strictly smaller diameter? This question stated as the affirmative has become known as Borsuk's conjecture. Although a series of results confirming Borsuk's conjecture had been obtained, Kahn and Kalai disproved this conjecture in 1993 in a space of very high dimension [6].

It is known that Borsuk's conjecture is true in dimensions $d \leq 3$. For the case $d = 3$, there exist several different proofs of the conjecture and, in particular, several different proofs for finite sets of points [7], [8]. They show that diameter graphs and Borsuk's problem for finite sets of points are closely related. Borsuk's conjecture for finite sets of points can be stated as follows: for any diameter graph G in $\mathbb{R}^d$, the inequality $\chi(G) \leq d + 1$ holds, where $\chi(G)$ is the chromatic number of the graph.

Apparently, the first work in which properties of finite diameter graphs were studied was the paper [9], whose authors proved that the number of edges in an n-vertex plane graph is at most n. This result implies, in particular, Borsuk's conjecture for $d = 2$. Vászonyi conjectured that a diameter graph in $\mathbb{R}^3$ on n vertices cannot contain more than $2n - 2$ edges. This conjecture was is proved independently by three authors, Grünbaum [10], Heppes [11], and Straszewicz [12]. In higher dimensions, the number of edges ceases to grow linearly; already in dimension 4, it is easy to construct a graph on n vertices in which the number of edges is of order n^2. However, the result of [9] has been generalized to higher dimensions in the form of the following conjecture.

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Conjecture (Schur [13]). *The maximum number of complete subgraphs of size d (d-cliques) in a diameter graph on n vertices in $\mathbb{R}^d$ equals n.*

We see that the case $d = 2$ corresponds to the result of [9]. In [13], Schur's conjecture was proved for $d = 3$. In the same paper, the following theorem was also proved.

Theorem 1. *Any diameter graph in $\mathbb{R}^d$ contains at most one $(d+1)$-clique.*

Note that a k-clique in a diameter graph corresponds to the vertices of a regular $(k-1)$-simplex $\Delta^{(k-1)}$ with edge length 1 in space.

Further progress in the study of Schur's conjecture was made by Morić and Pach.

Theorem 2 ([14]). *If any two d-cliques in a diameter graph on n vertices in $\mathbb{R}^d$ have at least $d-2$ common vertices, then the number of d-cliques in this graph is at most n.*

1.2. New Results

The main result of this paper is the following theorem.

Theorem 3. *Schur's conjecture is true in $\mathbb{R}^4$.*

The proof of Theorem 3 relies on Theorem 4, which is of independent interest.

Theorem 4. *Let $G = (V, E)$ be a diameter graph in which V is a subset of a 3-sphere S_r^3 of radius $r > 1/\sqrt{2}$. Then any two triangles in G have a common vertex.*

Slightly modifying the proof of Theorem 2, we can obtain its analog for spheres of sufficiently large radius.

Theorem 5. *Let G be an n-vertex diameter graph on a d-sphere S_r^d of radius $r > 1/\sqrt{2}$ in which any two d-cliques have at least $d-2$ common vertices. Then the total number of d-cliques in G is at most n. In other words, Schur's conjecture is true on the sphere for graphs of the specified form.*

It is easy to derive the following assertion from Theorems 4 and 5.

Corollary 1. *Schur's conjecture is true for diameter graphs on S_r^3, $r > 1/\sqrt{2}$.*

This paper also contains a simple proof of the fact that any two triangles in a diameter graph in $\mathbb{R}^3$ have a common vertex. In combination with Theorem 2, this gives a shorter proof of Schur's conjecture in $\mathbb{R}^3$.

In the next section, we prove Theorems 4 and 5. In Section 3, we prove Schur's conjecture in $\mathbb{R}^4$. In Section 4, we discuss Schur's conjecture in $\mathbb{R}^3$.

2. DIAMETER GRAPHS ON SPHERES

2.1. Proof of Theorem 4

In what follows, for each hyperplane π, we use π^+ and π^- to denote the half-spaces determined by π. First, we prove two auxiliary lemmas.

Lemma 1. *Suppose that, on the circular boundary of a half-disk W centered at a point O, any three points Q_1, A, and Q_2 are chosen so that A lies on the arc Q_1Q_2. Suppose also B is a point in W not coinciding with O. Then $|AB| < \max\{|BQ_1|, |BQ_2|\}$.*

Proof. The angle BOA is smaller than one of the angles BOQ_1 and BOQ_2. Without loss of generality, we assume that BOA is smaller than BOQ_2. Then the cosine theorem and the relation $|OQ_2| = |OA|$ imply $|BQ_2| > |BA|$. □

Lemma 2. *Let $B_1, \dots, B_k$ be balls of radii $r_1, \dots, r_k$ centered at $v_1, \dots, v_k$ and bounded by spheres $S_1, \dots, S_k$. Suppose that $S = \bigcap_{i=1}^k S_i$ is a sphere centered at v bounding a ball B in a plane π. Suppose also that v belongs to the convex hull of the points v_i. Then the projection of $\bigcap_{i=1}^k B_i$ on the plane π is contained entirely inside the ball B.*

Proof. Suppose that, on the contrary, there exists a point $x \in \bigcap_{i=1}^k B_i$ whose projection x' on the plane π is not inside B. Then, of course, we have $\|x' - v_i\| > r_i$ for each i. Consider the plane $\pi^\perp$ orthogonal to π and passing through v (and, naturally, through v_i, $i = 1, \dots, k$). Let u be the projection of x on the plane $\pi^\perp$, and let $\pi_u \subset \pi^\perp$ be the plane orthogonal to the vector $\overline{vu}$ and passing through v. Since v belongs to the convex hull of the points v_i, it follows that some v_i belongs to the closed half-space bounded by π_u that does not contain u. We have $(\overline{vv_i}, \overline{vu}) \le 0$ and, hence, $(\overline{vv_i}, \overline{vx}) \le 0$. Therefore,

$$\|v_i - x\|^2 \ge \|v - v_i\|^2 + \|v - x\|^2 > \|v - v_i\|^2 + \|v - x'\|^2 = \|x' - v_i\|^2 > r_i^2.$$

This contradicts the condition $x \in \bigcap_{i=1}^k B_i$. □

Suppose that G lies on a 3-sphere S of radius $r > 1/\sqrt{2}$ centered at v. The points $v_1, v_2, v_3 \in V$ form a triangle, i.e.,

$$|v_1 - v_2| = |v_2 - v_3| = |v_3 - v_1| = 1.$$

Consider the three unit balls B_1, B_2, and B_3 bounded by spheres S_1, S_2, and S_3 centered at v_1, v_2, and v_3. Let v_4 be one of the intersection points of the spheres S, S_1, S_2, and S_3.

We denote the hyperplane passing through v, v_1, v_2, and v_3 by π and the half-space determined by π and containing v_4 by π^+. By Δ we denote the intersection $S \cap B_1 \cap B_2 \cap B_3 \cap \pi^+$.

The following lemma plays a key role in the proof of Theorem 4.

Lemma 3. *If A and C are two points such that $A, C \in \Delta$ and $A, C \notin \{v_1, v_2, v_3, v_4\}$, then there exists a point $u \in \{v_1, v_2, v_3, v_4\}$ for which $\|A - u\| > \|A - C\|$.*

Proof. We prove the lemma by "moving" the point C so as to increase the distance between A and C.

Suppose that $C \notin \pi$. Then C belongs to the intersection Ω of several spheres among S, S_1, S_2, and S_3 (at least, it belongs to the sphere S). Let π_Ω denote the minimal plane containing Ω. Since the centers of the spheres S, S_1, S_2, and S_3 lie in π, it follows that $\pi_\Omega \perp \pi$ (or $\pi_\Omega = \pi$). Note that π_Ω is at least two-dimensional. Consider a two-dimensional plane $\pi' \subseteq \pi_\Omega$ such that the center of the sphere Ω and the point C belong to π' and $\pi' \perp \pi$. Let A' be the projection of A on π'. Since $\pi' \perp \pi$, we have $A' \in \pi^+$. Moreover, by virtue of Lemma 2, the projection of A falls inside Ω. Therefore, by Lemma 1, we can replace C by a point in the intersection of a larger number of spheres among S, S_1, S_2, and S_3 or by a point in the plane π so that the distance from A' to the new point is greater than $\|A' - C\|$. Therefore, the distance from the point A to this point is greater than $\|A - C\|$. In this way, we can move C to the point v_4 or to the plane π so that the distance $\|A - C\|$ increases.

It remains to consider the last case $C \in \pi$.

Let us embed space $\mathbb{R}^4$, which contains all of the points v, v_1, v_2, v_3, v_4, A, and C, in $\mathbb{R}^5$ so that the embedded space coincides with the hyperplane

$$\frac{x_1}{a} + x_2 + x_3 + x_4 + x_5 = 1 \tag{1}$$

and the points v, v_1, v_2, v_3, and v_4 have coordinates $(a,0,0,0,0)$, $(0,1,0,0,0)$, $(0,0,1,0,0)$, $(0,0,0,1,0)$, and $(0,0,0,0,1)$, respectively. Here $a > 0$. It is easy to show that this is possible, because the squared radius of the sphere S is a^2+1 and the squared radius of each of the spheres S_1, S_2, and S_3 is 2. The point A lies on the sphere S and inside the spheres S_i, $i \in \{1,2,3\}$, i.e., its coordinates (x_1,x_2,x_3,x_4,x_5) satisfy the equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - 2ax_1 = 1 \tag{2}$$

and the inequalities

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - 2x_j \le 1, \qquad j \in \{2,3,4\}.$$

This implies, in particular, that $ax_1 \le x_j$, $j \in \{2,3,4\}$. The point C belongs to the plane π and the sphere S.

Consider the projection A' of A on the plane π. This projection is inside the sphere S, because the plane π is diametral. If the distance from A' to one of the points v_1, v_2, and v_3 is greater than that from A' to C, then the distance from the point A itself to one of the points v_1, v_2, and v_3 is greater than that from A to C.

From now on, we work in the 3-plane π. Let π_0 denote the plane passing through v_1, v_2, and v_3; we denote the closed half-space containing v by π_0^+ and the other closed half-space by π_0^-. In a similar way, we define π_i, π_i^+, and π_i^- for $i \in \{1,2,3\}$.

It is easy to show that the point A' has coordinates

$$\left(x_1 + \frac{ax_5}{3a^2+1}\,,\ x_2 + \frac{a^2x_5}{3a^2+1}\,,\ x_3 + \frac{a^2x_5}{3a^2+1}\,,\ x_4 + \frac{a^2x_5}{3a^2+1}\,,\ 0\right).$$

If $x_1 + ax_5/(3a^2+1) \ge 0$, then $x_j + a^2x_5/(3a^2+1) \ge 0$, $j \in \{2,3,4\}$. Therefore, if $A' \in \pi_0^+$, then $A' \in \pi_i^+$, $i \in \{1,2,3\}$.

If $x_1 + ax_5/(3a^2+1) < 0$, then at most one of the three inequalities $x_j + a^2x_5/(3a^2+1) < 0$, $j \in \{2,3,4\}$, holds. Indeed, suppose that, on the contrary, two such inequalities hold (we can assume without loss of generality that these are the inequalities with $j = 2,3$). Relation (1) implies

$$x_4 + \frac{a^2x_5}{3a^2+1} > 1,$$

and since A' lies inside the sphere S, it follows that

$$\left(x_1 + \frac{ax_5}{3a^2+1}\right)^2 + \left(x_2 + \frac{a^2x_5}{3a^2+1}\right)^2 + \left(x_3 + \frac{a^2x_5}{3a^2+1}\right)^2 + \left(x_4 + \frac{a^2x_5}{3a^2+1}\right)^2 \\ - 2a\left(x_1 + \frac{ax_5}{3a^2+1}\right) \le 1.$$

Therefore, $x_4 + a^2x_5/(3a^2+1) < 1$. We have obtained a contradiction.

Thus, in any case, A' belongs to two of the three half-spaces π_i^+, $i \in \{1,2,3\}$.

Consider the following two possible cases for the location of C.

I. Suppose that C belongs only to the sphere S and does not belong to the spheres S_1, S_2, and S_3. Let us draw a plane through v, A', and C. The plane $x_1 = a$ is supporting for the possible loci of A'; therefore, using Lemma 1, we can replace C by a point in the intersection of one of the spheres S_i and S so that the distance from A' to C increases. Thus, this case reduces to the second one.

II. Suppose that C lies in the intersection Ω of two spheres, S and, say, S_1 (the intersection of a larger number of spheres is already a vertex of the simplex). Consider the projection of A' on the minimal plane

π_Ω containing Ω. Note that either $A' \in \pi_2^+$ or $A' \in \pi_3^+$. We can assume without loss of generality that $A' \in \pi_2^+$. Then the projection of A' on π_Ω belongs to π_2^+, because $\pi_2 \perp \pi_\Omega$. Again applying Lemma 1, we see that the distance from the projection of A' (and, therefore, from A) to C is smaller than the distance to v_2 or v_3. □

Now, we can prove Theorem 4. Take any two triangles v_1, v_2, v_3 and w_1, w_2, w_3. Consider the plane π passing through the center of the sphere S and the points v_1, v_2, and v_3. One of the two half-spaces determined by π contains at least two points among the w_j. If none of them coincides with the vertices v_i, then we obtain a contradiction with Lemma 3. This proves Theorem 4.

2.2. Proof of Theorem 5

The proof of this theorem in essentially identical to the proof of Theorem 2, and we do not reproduce it. The main part of the proof is combinatorial and does not require any changes. The only thing which must be verified is the following assertion.

Lemma 4. *Let $S = S_r^d$ be a d-sphere of radius $r \geq 1/\sqrt{2}$, and let Δ be a unit simplex on k vertices $v_1, \dots, v_k$ which lie on the sphere S. Then the intersection Ω of the sphere S with the unit spheres centered at $v_1, \dots, v_k$ is a sphere of radius $r_\Omega \geq 1/\sqrt{2}$.*

Proof. Let $v = 1/k \sum_{i=1}^k v_i$ be the center of the sphere circumscribed about Δ. By Jung's theorem, the squared radius of this sphere equals $(k-1)/2k$. Let s be the center of S. Obviously, the center w of Ω belongs to the segment vs; we set

$$b^2 \overset{\text{def}}{=} \|v - s\|^2 = r^2 - \frac{k-1}{2k}.$$

Consider a partition of this segment into pieces of lengths a and $b - a$. We have

$$r_\Omega^2 = r^2 - (b-a)^2 = \frac{k+1}{2k} - a^2.$$

This implies

$$2r_\Omega^2 = r^2 - b^2 + \frac{k+1}{2k} + 2ab - 2a^2 = 1 + 2a(b-a) \geq 1. \quad \square$$

It follows from Lemma 4 that all the radii of the spheres obtained in adapting the proof of Theorem 2 to the case under consideration are at least $1/\sqrt{2}$, as in the case of $\mathbb{R}^d$; therefore, we can use Lemma 2.2 from the paper [14], and the proof requires no essential changes. We quote Lemma 2.2 from [14] below.

Lemma 5. *Suppose given a diameter graph on a sphere S_r^2, where $r \geq 1/\sqrt{2}$. Then the number of edges in this graph does not exceed the number of vertices.*

3. COMPLETION OF THE PROOF OF SCHUR'S CONJECTURE

Let G be a diameter graph. The following lemma is valid.

Lemma 6. *If a diameter graph G in $\mathbb{R}^4$ contains two cliques K^1 and K^2 of size 4, then these cliques cannot intersect in one vertex.*

Proof. Suppose that K^1 and K^2 have a common vertex v. Then the remaining vertices of these two cliques lie on the unit 3-sphere S centered at v. If K^1 and K^2 have no other common vertices, then the vertices of K^i form two unit triangles on S. This contradicts Theorem 4 and the fact that G is a diameter graph. Thus, K^1 and K^2 intersect in at least two vertices. □

We also need the following lemma, which was proved in [15].

Lemma 7. *If a, b, c, and d are four points on a 2-sphere of radius at least* $1/\sqrt{2}$*, the diameter of the set of these four points equals* 1*, and* $|ab| = |cd| = 1$*, then the minor diametral arcs ab and cd intersect.*

Thus, according to Lemma 6, any two cliques of size 4 either have no common vertices or intersect in at least two vertices. Let us show that the vertex set V of the graph can be represented as the union of disjoint sets $V_1, \dots, V_k$ of vertices with the following properties. First, any 4-clique in the graph is contained entirely in one of the sets $V_1, \dots, V_k$. Secondly, in each of the sets V_i, all 4-cliques intersect in at least two vertices. In other words, we want to divide all 4-cliques into equivalence classes so that two clique are considered equivalent if they intersect. For V_i we take the union of all vertices of cliques in the ith equivalence class. The vertices not contained in any clique are included in any class.

To prove the possibility of such a partition, we must show that the equivalence relation is well defined. Let us check transitivity, i.e., prove that there is no triple of cliques K^1, K^2, K^3 for which

$$|K^1 \cap K^2| = |K^1 \cap K^3| = 2, \qquad |K^2 \cap K^3| = 0$$

(if K^1 intersects one of the other two cliques in more than two points, then, according to Dirichlet's principle, these two cliques intersect as well). Suppose that the clique K^1 has vertices v_1, v_2, v_3, and v_4, the vertices v_4, v_1, and v_2 belong to K^2, and v_3 and v_4 belong to K^3. Suppose also that $w_1, w_2 \in K^2$ and $w_3, w_4 \in K^3$. We denote the plane passing through v_1, v_2, v_3, and v_4 by π. The points v_1, v_2, w_3, and w_4 lie on a 2-sphere S_1 of radius $\sqrt{3}/2$ centered at the midpoint of the segment joining v_3 and v_4, and this sphere is contained in the plane orthogonal to this segment. Similarly, the points v_3, v_4, w_1, and w_2 lie on a 2-sphere S_2 of radius $\sqrt{3}/2$ centered at the midpoint of the segment joining the vertices v_1 and v_2, and this sphere is contained in the plane orthogonal to this segment.

By virtue of Lemma 7, the arcs v_3v_4 and w_1w_2 intersect (as well as the arcs v_1v_2 and w_3w_4). Note that one of the vertices w_1 and w_2, say w_1, may occur inside the arc v_3v_4. Since both arcs are shorter than the quarter of the great circle and are themselves arcs of great circles, the spherical cap centered at w_2 whose boundary contains w_1 cannot contain v_3 and v_4 simultaneously; i.e., either $\|w_2 - v_3\|$ or $\|w_2 - v_4\|$ is larger than 1. It follows that none of the points w_i can lie in the plane π, because, if such a point existed, it would coincide with one of the vertices v_i, and the intersection of one of the pairs of cliques would contain more than two points. We have already eliminated this case. Without loss of generality, we can assume that w_1 and w_3 belong to π^+ and w_2 and w_4 belong to π^-. Consider the unit 3-spheres S_1^w and S_2^w centered at w_1 and w_2. They intersect the sphere S_1 in the points v_1 and v_2, and none of the spheres S_i^w contains S_1 entirely, because the segments joining the center of S_1 with w_1 and w_2 are not orthogonal to the plane containing S_1. Indeed, for both segments, the components orthogonal to π are nonzero. However, any vector orthogonal to π is contained in the plane of the sphere S_1. On the other hand, the projection of the center of any sphere entirely containing S_1 on the minimal plane containing S_1 coincides with the center of S_1, so that the vector pointing from the center of S_1 to this center has zero component in the plane of S_1. Therefore, S_1 intersects the spheres S_i^w in circles S_i' on S_1 passing through the point v_1 and v_2 (see the figure).

We want to show that any arrangement of the points w_3 and w_4 leads to a contradiction with one of the conditions that follow from the fact that the graph under consideration is a diameter graph. The points w_3 and w_4 lie on the sphere S_1. Moreover, $w_3, w_4 \in B_1^w \cap B_2^w \cap B_1^v \cap B_2^v$, where the B_i^w are unit balls centered at the points w_i and the B_i^v are unit balls centered at the points v_i. We claim that the intersection $S_1 \cap B_1^w \cap B_2^w \cap B_1^v \cap B_2^v$ is empty.

In the rest of the proof, we work in the hyperplane containing S_1; thus, for the intersections of the 3-spheres S_1^w and S_2^w with this hyperplane we use the same notation as for these spheres themselves. Let w_1' and w_2' denote the projections of the corresponding centers; these are the centers of the 2-spheres S_1^w and S_2^w. Note that w_1' lies in π^+ (here and in what follows, by π we mean the intersection of π and the hyperplane under consideration; the half-spaces are assumed to be open) and w_2' lies in π^-. Let π_1 be the plane orthogonal to the segment v_1v_2 and passing through the midpoint of this segment. This plane contains the center of S_1 and the points w_1' and w_2'. We assume that v_1 lies in π_1^+ and v_2 lies in π_1^-. Let π_2 be the 2-plane orthogonal to π and π_1 and passing through the center of S_1. It is easy to show that the intersection $S_1 \cap B_1^v \cap B_2^v$ is entirely contained in the open half-space bounded by the plane π_2. Indeed,

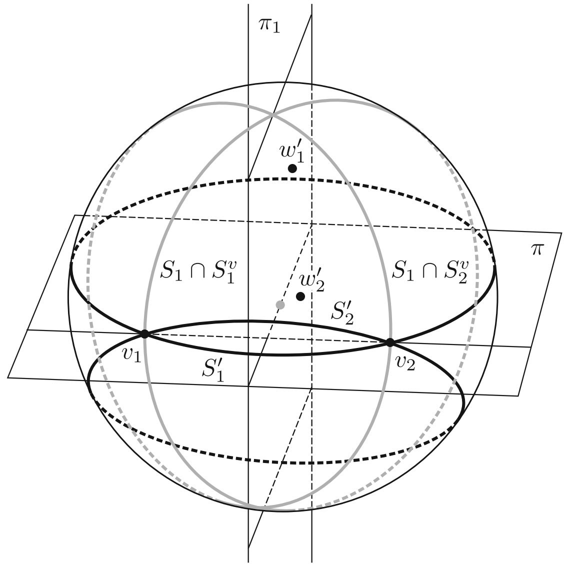

Figure.

any point on S_1 belonging to π_2^- forms an angle larger than $\pi/2$ with one of the points v_1 and v_2 and the center of S_1; since the radius of S_1 is larger than $1/\sqrt{2}$, it follows that the distance between these points is larger than 1.

Let us show that the circles S'_1 and $S_1 \cap S_1^v$ inside the half-space π_1^- intersect in only one point, v_2. Since S'_1 lies on S_1, it follows that the intersection of these two circles coincides with the intersection of the sphere S_1^v and the circle S'_1. The center of S_1^v lies on S'_1; therefore, the intersection points of these two spheres must be symmetric with respect to the line joining their centers in the plane containing S'_1. But both centers lie in π_1^+; hence at least one of the intersection points must lie in π_1^+. A similar assertion is valid for the circles S'_2 and $S_1 \cap S_1^v$ and, in the symmetric half-space π_1^+, for the circles $S_1 \cap S_2^v$ and S'_i.

The domain $S_1 \cap B_1^w$ is located above the plane of the circle S'_1 (on the side to which the normal to the plane π directed to π^+ points; in other words, this domain contains the intersection point of the ray Ow'_1 with the sphere S_1). Similarly, $S_1 \cap B_2^w$ is located above the plane of the circle S'_2. Note also that the plane of any of the circles S'_i, $i = 1, 2$, cannot be parallel to π_2, because, if it were, then the point w'_i would belong to the plane π. It follows readily that, depending on the arrangement of w'_1 and w'_2, there are two possible locations of the domain $S_1 \cap B_1^w \cap B_2^w$.

Case 1. The domain $S_1 \cap B_1^w \cap B_2^w$ on the sphere is bounded by the major arcs of the circles S'_i with endpoints v_1 and v_2. On the other hand, the domain $S_1 \cap B_1^v \cap B_2^v$ is determined by

$$S_1 \cap B_1^v \cap B_2^v = S_1 \cap B_1^v \cap \pi_1^- \bigcup S_1 \cap B_2^v \cap \pi_1^+.$$

It is clear from the above considerations that the domains $S_1 \cap B_1^w \cap B_2^w$ and $S_1 \cap B_2^v \cap \pi_1^+$ intersect only in the vertex v_1, and the domains $S_1 \cap B_1^w \cap B_2^w$ and $S_1 \cap B_1^v \cap \pi_1^-$ intersect only in v_2. Thus,

$$S_1 \cap B_1^v \cap B_2^v \bigcap S_1 \cap B_1^w \cap B_2^w = \{v_1, v_2\},$$

and there is nowhere to place the points w_3 and w_4.

Case 2. The domain $S_1 \cap B_1^w \cap B_2^w$ on the sphere is bounded by the minor arcs of the circles S'_i with endpoints v_1 and v_2. In this case, the domain $S_1 \cap B_1^w \cap B_2^w$ is entirely contained in the spherical cap H cut off by the plane π' parallel to π_2 and passing through v_1 and v_2. Moreover, the intersection of $S_1 \cap B_1^w \cap B_2^w$ and π' consists only of the points v_1 and v_2. Indeed, $S_1 \cap B_1^w$ intersects $\pi' \cap S_1 \cap \pi^-$ only in v_1 and v_2 (because $S_1 \cap B_1^w$ is located as described in the paragraph before Case 1). Similarly, $S_1 \cap B_2^w$ intersects $\pi' \cap S_1 \cap \pi^+$ only in v_1 and v_2. On the other hand, the minor arcs of the circles S'_i must lie inside H.

The circle $S_1 \cap \pi'$ has diameter 1; therefore, points lying on the sphere S_1 strictly inside H cannot be a distance 1 apart. Thus, the distance between points of $S_1 \cap B_1^w \cap B_2^w$ cannot equal 1, unless these points coincide with v_1 and v_2, and hence there is no place for the points w_3 and w_4 inside $S_1 \cap B_1^w \cap B_2^w$.

We have shown that the 4-cliques are indeed divided into equivalence classes; thus, the partition of the vertex set V into subsets $V_1, \dots, V_k$ described at the beginning of the proof exists. Applying Theorem 2 to each V_i, we see that the number of 4-cliques in each of the sets V_i is at most $|V_i|$ and, therefore, the total number of cliques is at most $\sum_i |V_i| = n$. This completes the proof of Schur's conjecture in $\mathbb{R}^4$.

Remark. We have proved Schur's conjecture in $\mathbb{R}^4$ without proving Conjecture 3 from [14], which asserts that any two 4-cliques in a diameter graph in $\mathbb{R}^4$ must have at least two common vertices. This conjecture is very likely to be true, but it is of auxiliary character, and proving it is beyond our purposes in this paper.

4. SCHUR'S CONJECTURE IN $\mathbb{R}^3$

In this section, we give a short proof of the following theorem (which is a special case of Conjecture 3 in [14]).

Theorem 6. *Any two triangles in any diameter graph G in $\mathbb{R}^3$ have a common vertex.*

This theorem combined with Theorem 2 proves Schur's conjecture in $\mathbb{R}^3$.

Definition 1. The *Reuleaux simplex of dimension d* is the solid in the intersection of unit balls $B_1^d(x_i)$ centered at points x_i, where the x_i, $i = 1, 2, \dots, d+1$, are the vertices of the unit simplex Δ^d. The *Reuleaux tetrahedron* is the Reuleaux simplex of dimension $d = 3$.

The term arose from the analogy with the *Reuleaux triangle*, which is the plane figure obtained in the case $d = 2$. Note that, unlike the Reuleaux triangle, the Reuleaux simplex of dimension $d \geq 3$ is not a solid of constant width.

The key role is played by the following lemma, which is an analog of Lemma 3.

Lemma 8. *In space $\mathbb{R}^3$, consider the intersection Δ of the Reuleaux tetrahedron with vertices $v_1, \dots, v_4$ and the half-space π^+ containing v_4 and determined by the plane π passing through v_1, v_2, and v_3. Let $X, Y \in \Delta$ be any points not coinciding with the vertices and such that $|X - Y| = a$. Then there exists an i for which $|X - v_i| > a$.*

Proof. Suppose that both points belong to a plane face of Δ (this face is a Reuleaux triangle). Without loss of generality, we can assume that Y lies on the boundary of the Reuleaux triangle and on a circle centered at v_1. Consider the closed half-plane determined by the line Yv_1. The distance between X and Y is smaller than that between X and the vertex of the Reuleaux triangle contained in the half-plane not containing X.

Now, suppose that the points do not belong to the plane π. If Y is nearer to π, then, without loss of generality, we can assume that Y coincides with its projection on the plane π. Let X' be the projection of X on this plane. From the above considerations, we find a vertex v_i for which $X'Y < X'v_i$ and, applying the Pythagorean theorem, obtain $XY < Xv_i$.

Now, suppose that the point X is nearer to π. Let us extend the segment XY to a segment XY' so that the endpoint Y' belongs to the boundary of Δ. We must consider two cases.

Case 1. The point Y' belongs to the sphere centered at v_1 and does not belong to the spheres centered at the other vertices of the tetrahedron. We denote the spherical part of the boundary of the truncated Reuleaux tetrahedron centered at v_1 by s. Consider the plane p'' containing the points X, Y', and v_1. Note that Y' lies on the arc D in the intersection $p'' \cap s$.

Suppose that the arc D intersects the boundary of s in points Q_1 and Q_2. We apply Lemma 1 to the points Q_1 and Q_2 (playing the role of themselves), to X and Y' (as B and A), and to v_1 (as O). Note that all these points are indeed inside the required half-disk, because v_1 is a boundary point of the Reuleaux tetrahedron, and we can draw a (support) plane through this point so that all points under consideration belong to the same half-space with respect to this plane. The intersection of this half-space with the disk centered at v_1 is the required half-disk. We see that $|Q_1X|$ or $|Q_2X|$ is larger than $|XY'|$.

Thus, this case reduces to the following one.

Case 2. Suppose that the point Y' lies on the arc l in the intersection of the two spherical parts of the boundary formed, say, by the spheres centered at v_1 and v_2. The center of the circle containing l is the midpoint v of the segment v_1v_2. Let X' be the projection of X on the plane containing the arc l. To complete the proof, it suffices to show that $X'v_3$ or $X'v_4$ is larger than $X'Y'$. After this, it only remains to apply the Pythagorean theorem.

The required assertion again follows by Lemma 1. The choice of the points is clear, and for the plane cutting off a half-disk we take π. This completes the proof of Lemma 8. □

Lemma 8 with $a = 1$ implies that Δ cannot contain two points a unit distance from each other none of which coincides with a vertex of the tetrahedron.

Consider a diameter graph G in $\mathbb{R}^3$ and a triangle $V = v_1v_2v_3$ in G. All vertices of G must lie in the union of two domains of the form Δ (these are the two domains constructed from the vertices v_1, v_2, and v_3 and the plane π containing these vertices).

Consider a triangle $W = w_1w_2w_3$ in G. One of the above-mentioned domains of the form Δ contains two vertices of W. Let v_4 denote the fourth vertex of Δ. By virtue of Lemma 8, either W must share one vertex with the triangle V, or v_4 is one of the vertices of W. In the latter case, it is easy to see that, inside Δ, only the points v_1, v_2, and v_3 are at distance 1 from v_4, and, therefore, the second vertex of W contained inside Δ coincides with a vertex of V. This completes the proof of the theorem.

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