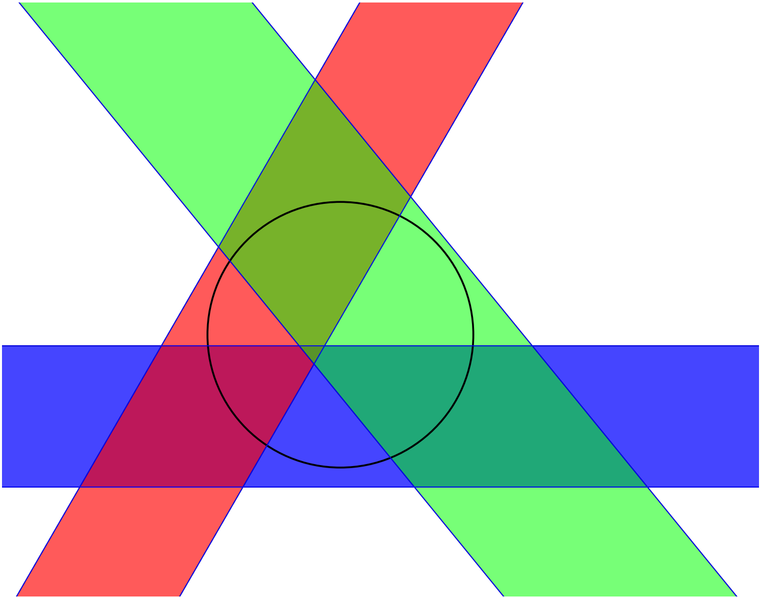
Covering the space by slabs

Andrey B. Kupavskii, Janos Pach



Slabs

The set of points S lying between two parallel hyperplanes in $\mathbb{R}^d$ at distance w from each other is called a *slab* of *width* w.

Makai and Pach, Erdős and Straus, Groemer:
There is a constant c such that any system of slabs in the plane with total width at least c permits a translative covering of a disk of diameter 1.

Slabs

The set of points S lying between two parallel hyperplanes in $\mathbb{R}^d$ at distance w from each other is called a *slab* of *width* w.

Makai and Pach, Erdős and Straus, Groemer:
There is a constant c such that any system of slabs in the plane with total width at least c permits a translative covering of a disk of diameter 1.

It implies that ny sequence of slabs whose total weight is divergent permits a translative covering of the whole plane.

What about higher dimensions?

It implies that ny sequence of slabs whose total weight is divergent permits a translative covering of the whole plane.

What about higher dimensions?

Makai-Pach conjecture

Conjecture (Makai-Pach)

Let d be a positive integer. A sequence of slabs in $\mathbb{R}^d$ with widths $w_1, w_2, \ldots$ permits a translative covering of $\mathbb{R}^d$ if and only if $\sum_{i=1}^{\infty} w_i = \infty$.

Groemer: It is true if $\sum_{i=1}^{\infty} w_i^{\frac{d+1}{2}} = \infty$.

Theorem 1 (Kupavskii-Pach)

It is true if $w_1 \geqslant w_2 \geqslant \ldots$ is a monotone decreasing infinite sequence of positive numbers such that

$$\limsup_{n\to\infty} \frac{w_1 + w_2 + \ldots + w_n}{\log(1/w_n)} > 0.$$

Makai-Pach conjecture

Conjecture (Makai-Pach)

Let d be a positive integer. A sequence of slabs in $\mathbb{R}^d$ with widths $w_1, w_2, \ldots$ permits a translative covering of $\mathbb{R}^d$ if and only if $\sum_{i=1}^{\infty} w_i = \infty$.

Groemer: It is true if $\sum_{i=1}^{\infty} w_i^{\frac{d+1}{2}} = \infty$.

Theorem 1 (Kupavskii-Pach)

It is true if $w_1 \geqslant w_2 \geqslant \ldots$ is a monotone decreasing infinite sequence of positive numbers such that

$$\limsup_{n\to\infty} \frac{w_1 + w_2 + \ldots + w_n}{\log(1/w_n)} > 0.$$

Makai-Pach conjecture

Conjecture (Makai-Pach)

Let d be a positive integer. A sequence of slabs in $\mathbb{R}^d$ with widths $w_1, w_2, \ldots$ permits a translative covering of $\mathbb{R}^d$ if and only if $\sum_{i=1}^{\infty} w_i = \infty$.

Groemer: It is true if $\sum_{i=1}^{\infty} w_i^{\frac{d+1}{2}} = \infty$.

Theorem 1 (Kupavskii-Pach)

It is true if $w_1 \geqslant w_2 \geqslant \ldots$ is a monotone decreasing infinite sequence of positive numbers such that

$$\limsup_{n\to\infty} \frac{w_1 + w_2 + \ldots + w_n}{\log(1/w_n)} > 0.$$

Proof of theorem 1

Theorem 1'

Let d be a positive integer, and let $w_1 \geqslant w_2 \geqslant \ldots \geqslant w_n$ be positive numbers such that

$$w_1 + w_2 + \ldots + w_n \geqslant 3d \log(2/w_n).$$

Then any sequence of slabs $S_1, \ldots, S_n \subset \mathbb{R}^d$ with widths $w_1, \ldots, w_n$, resp., permits a translative covering of a d-dimensional ball of diameter $1 - w_n/2$.

Controlling polynomials

Let $\mathcal{F}$ be a class of real functions $\mathbb{R} \to \mathbb{R}$. We say that a sequence of positive numbers $x_1, x_2, \ldots$ is *$\mathcal{F}$-controlling* if there exist reals $y_1, y_2, \ldots$ with the property that for every $\ell \in \mathcal{P}$, one can find an i with

$$|f(x_i) - y_i| \leqslant 1.$$

Let $\mathcal{P}_d$ denote the class of polynomials $\mathbb{R} \to \mathbb{R}$ of degree at most d.

Theorem 2 (Kupavskii-Pach)

Let d be a positive integer and $x_1 \leqslant x_2 \leqslant \ldots$ be a monotone increasing infinite sequence of positive numbers. The sequence $x_1, x_2, \ldots$ is *$\mathcal{P}_d$-controlling* if and only if

$$\lim_{n\to\infty} (x_1^{-d} + x_2^{-d} + \ldots + x_n^{-d}) = \infty.$$

Controlling polynomials

Let $\mathcal{F}$ be a class of real functions $\mathbb{R} \to \mathbb{R}$. We say that a sequence of positive numbers $x_1, x_2, \ldots$ is *$\mathcal{F}$-controlling* if there exist reals $y_1, y_2, \ldots$ with the property that for every $\ell \in \mathcal{P}$, one can find an i with

$$|f(x_i) - y_i| \leqslant 1.$$

Let $\mathcal{P}_d$ denote the class of polynomials $\mathbb{R} \to \mathbb{R}$ of degree at most d.

Theorem 2 (Kupavskii-Pach)

Let d be a positive integer and $x_1 \leqslant x_2 \leqslant \ldots$ be a monotone increasing infinite sequence of positive numbers. The sequence $x_1, x_2, \ldots$ is *$\mathcal{P}_d$-controlling* if and only if

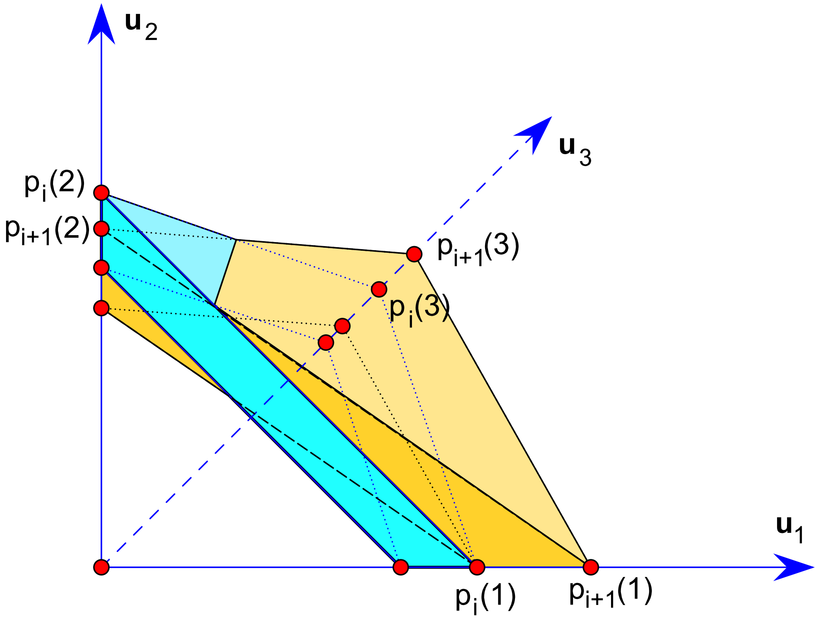
$$\lim_{n\to\infty} (x_1^{-d} + x_2^{-d} + \ldots + x_n^{-d}) = \infty.$$

For every positive integer d, for any system of $d+1$ linearly independent vectors $\mathbf{u}_1, \dots, \mathbf{u}_{d+1}$ in $\mathbb{R}^{d+1}$, and for any $\gamma > 0$, there is a constant c with the following property.
Given any system of slabs S_i $(i = 1, \dots, n)$ in $\mathbb{R}^{d+1}$, whose normal vectors $\mathbf{x}_i$ satisfy the conditions

$$(i) \qquad \frac{\langle \mathbf{x}_{i+1}, \mathbf{u}_1 \rangle}{\langle \mathbf{x}_i, \mathbf{u}_1 \rangle} \leqslant \frac{\langle \mathbf{x}_{i+1}, \mathbf{u}_j \rangle}{\langle \mathbf{x}_i, \mathbf{u}_j \rangle},$$

$$(ii) \qquad \langle \mathbf{x}_i, \mathbf{u}_j \rangle \geqslant \gamma \|\mathbf{x}_i\| \|\mathbf{u}_j\|$$

for every i and j, and whose total width $\sum_{i=1}^{n} w_i$ is at least c, the slabs S_i permit a translative covering of a $(d+1)$-dimensional ball of unit diameter.



u_2
u_3
$p_i(2)$
$p_{i+1}(2)$
$p_{i+1}(3)$
$p_i(3)$
u_1
$p_i(1)$
$p_{i+1}(1)$