Proof of Schur's conjecture

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01.02.2014
Moscow Workshop on Combinatorics and Number Theory

Diameter graphs

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A graph $G = (V, E)$ is a **diameter graph** in $\mathbb{R}^d$ (on S_r^d), if $V \subset \mathbb{R}^d$ (S_r^d) is a finite set of diameter 1, and edges of G are formed by vertices that are at unit distance apart.

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Conjecture (Schur et.al., 2003)

Any diameter graph G on n vertices in $\mathbb{R}^d$ has at most n d-cliques.

Version for spheres: Any diameter graph G on n vertices on the sphere S_r^d with $r > 1/\sqrt{2}$ has at most n d-cliques.

Previous results

Schur's conjecture holds for $d = 2$ (Hopf, Pannwitz, 1934), $d = 3$ (Schur et. al., 2003), $d = 4$ (Kupavskii, 2013)

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The Morić-Pach conjecture, 2013

Any two d-cliques in a diameter graph in $\mathbb{R}^d$ share at least $d - 2$ vertices.

Theorem 1 (Kupavskii, Polyanskii, 2013)

Schur's conjecture and the Morić-Pach conjecture hold
1. In the space $\mathbb{R}^d$.
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The proof is based on induction and the following

Theorem 2 (Kupavskii, Polyanskii, 2013)

Consider a diameter graph G
1. In the space $\mathbb{R}^d$, $d \geq 3$.
2. On the sphere S_r^d of radius $r > 1/\sqrt{2}$, $d \geq 3$.

Then any two d-cliques in G must share a vertex.

A rugby ball

A **rugby ball** Θ in $\mathbb{R}^d$ is a set formed by the intersection of the balls $B_i = B_1^d(v_i)$ of unit radius with centers in $v_i,\ i = 1, \ldots, d$, where v_i are the vertices of a unit d-simplex in $\mathbb{R}^d$.

Proof of Theorem 2. Step 1

- Consider a diameter graph G and two d-cliques K_1, K_2 in G. Denote by $v_1, \ldots, v_d$ the vertices of K_1. Denote by $u_1, \ldots, u_d$ the vertices of K_2.
- Form a rugby ball Θ on K_1 and denote by π the hyperplane containing K_1.
- Put $S_i = S_1^{d-1}(v_i)$ (S_i bounds the balls B_i)
- By v_{d+1} denote one of the two points that lie in the set formed by the intersection of S_i.
- By π^+ denote the half-space that is determined by π and contains v_{d+1}. By π^- denote the other half-space that is determined by π.
- Put $\Delta^+ = \Theta \cap \pi^+$, $\Delta^- = \Theta \cap \pi^-$.

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- $S \cap S_i$ is a sphere that lies in the hyperplane π_i, which is orthogonal to π.
- By π_i^+ denote the open half-space that is determined by π_i and contains v_i.

property

$a \in \Theta \backslash B \Rightarrow$ the projection a' of a on the plane π falls strictly inside $T = \text{conv}(v_1, \ldots, v_d)$.

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Lemma

Suppose that:

- $a, b \in \Delta^+$;
- the projection a' of a falls into $T = \mathrm{conv}(v_1, v_2, \ldots, v_d)$;
- $\|a - b\| = 1$.

Then we have two possibilities:

1. a or $b \in \{v_1, \ldots, v_d\}$;
2. $b \in \pi \cap \Theta$ and $a' \in \partial T$.

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corollary

Suppose that there are at least two vertices a, b of K_2 in Δ^+. If $a \notin B$, then $b \in \{v_1, \ldots, v_d\}$ (and thus Theorem 2 holds!!!!).

Proof of Theorem 2. Step 2.

Now we are left with the following two possibilities.

case 1 On both sides of the plane π we have at least two points of K_2, or all vertices of K_2 lie on one side.

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- Suppose that all the vertices of K_2 lie inside B.
- The radius of the minimal ball that contains K_2 equals the radius of B. Then all the points of K_2 must lie on S.
  (by the lemma)$\Rightarrow$
- Some of the vertices of K_2 must coincide with some of the $v_1, \ldots, v_d$.

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We perturb the simplex K_1:

- Suppose the distance between u_1 and v_1 is strictly less than 1.
- We start to rotate v_1 around the vertices $v_2, \ldots, v_d$, which are fixed.

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- We stop the rotation procedure if one of the two following events happen:
  1. The distance between v' and w_1 is equal to 1.
  2. Some of the $u_2, \ldots, u_d$ fall on the plane π' that passes through $v', v_2, \ldots, v_d$.

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case 2 $u_1 \in \Delta^+$, $u_2, \ldots, u_d \in \Delta^-$

- We start to rotate v_1 around the vertices $v_2, \ldots, v_d$, which are fixed.
- We stop the rotation procedure if one of the two following events happen:
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  2. Some of the $u_2, \ldots, u_d$ fall on the plane π', which is a plane that passes through $v', v_2, \ldots, v_d$.

i If the first event happens then we change v_1 to v', take another vertex of K_1 and proceed by induction.

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ii The second event reduces (in a certain sense) to **case 1**.

All the vertices $v_1, \dots, v_d, u_2, \dots, u_d$ lie on a unit sphere with the center in $u_1 \Rightarrow$ (by the inductive assumption) one of the vertices $u_2, u_3, \dots, u_d$ coincide with one of the vertices K_1.

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Theorem 2 proved! $\Rightarrow$ Schur's conjecture holds!

Conjectures and questions

Conjecture (Morić and Pach)

Let $a_1 \ldots, a_d$ and $b_1 \ldots, b_d$ be two simplices on d vertices in $\mathbb{R}^d$ with $d \geq 3$, such that all their edges have length at least 1. Then there exist $i, j \in \{1, \ldots, d\}$ such that $\|a_i - b_j\| \geq 1$.

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In general, they asked the following question:

Problem (Morić and Pach)

For a given d, characterize all pairs k, l of integers such that for any set of k red and l blue points in $\mathbb{R}^d$ we can choose a red point r and a blue point b such that $\|r - b\|$ is at least as large as the smallest distance between two points of the same color.

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Conjecture (Kupavskii and Polyanskii)

Given two unit simplices in $\mathbb{R}^d$, one on $d+1$ vertices, the other on $\lfloor \frac{d+1}{2} \rfloor + 1$ vertices, either they share a vertex, or the diameter of their union is strictly larger than 1.