Diameter graphs in $\mathbb{R}^4$

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Introduction. Diameter graphs

Diameter graph

A graph $G = (V, E)$ is a *diameter graph in* $\mathbb{R}^d$ if $V \subset \mathbb{R}^d$, V is finite, $\operatorname{diam} V = 1$ and $E \subseteq \{(x, y), x, y \in \mathbb{R}^d, |x - y| = 1\}$, where $|x - y|$ denotes the Euclidean distance between x and y.

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- Vázsonyi's conjecture
- Extremal problems for distance and diameter graphs

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Borsuk's problem

1933, K. Borsuk:

Is it true that any set of diameter 1 in $\mathbb{R}^d$ can be partitioned into $d+1$ parts of smaller diameter?

- 1955, H. Eggleston, true for $d = 3$.
- 1993, J. Kahn, G. Kalai, false for $d = 1325, d \geqslant 2016$.
- 2013, A. Bondarenko, false for $d \geqslant 65$.

All the known counterexamples are based on finite sets of points.

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Finite version of Borsuk's problem. Vázsonyi's conjecture.

Borsuk's problem for finite sets translates into the question

Is it true that any diameter graph G in $\mathbb{R}^d$ satisfies $\chi(G) \leqslant d+1$?

- H. Hopf, E. Pannvitz: the number of edges in a diameter graph on n vertices in $\mathbb{R}^2$ is at most n.
- Vázsonyi's conjecture: the number of edges in a diameter graph on n vertices in $\mathbb{R}^3$ is at most $2n-2$.
- It was proved independently by B. Grünbaum, A. Heppes and S. Straszewicz.
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Diameter graphs on the spheres

We consider diameter graphs with **unit** edges on spheres of radius $r > 0$.

Any diameter graph G on n vertices on S_r^2 with $r > \sqrt{3/8}$ has at most n edges.

Theorem 1 (AK, 2013)

Let G be a diameter graph on n vertices on S_r^3. If $r > 1/\sqrt{2}$, then:

- G has at most $2n - 2$ edges.
- $\chi(G) \leqslant 4$.
- Any two odd cycles in G have a common vertex.

Remark. One idea important for the proof was communicated to me by A. Akopyan. Moreover, he told me that he proved this theorem (and even an analogue for Lobachevskiy space), but have never written it down.

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Discussion

- The proof of the theorem is based on the approach of V. Dol'nikov, which was later developed by K. Swanepoel.
- The bound on r in the theorem is tight — for arbitrary $n \in N$ one can realize $K_{n,n}$ as a diameter graph on $S^3_{1/\sqrt{2}}$.
- One can realize K_4 as a diameter graph on the sphere $S^2_{\sqrt{3/8}}$ and K_5 on the sphere $S^3_{\sqrt{2/5}}$.
- One can have two vertex-disjoint triangles in a diameter graph on $S^3_{1/\sqrt{3}}$.

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Number of edges in diameter and distance graphs

$D_d(l, n)$ — the maximum number of cliques of size l in a diameter graph on n vertices in $\mathbb{R}^d$.

$U_d(l, n)$ — the same for unit distance graphs in $\mathbb{R}^d$.

(A graph $G = (V, E)$ is a unit distance graph in $\mathbb{R}^d$, if $V \subset \mathbb{R}^d$ and E is formed by pairs of vertices that are at unit distance apart.)

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(P. Erdős) For $d \geqslant 4$ we have $U_d(2,n), D_d(2,n) \sim \frac{\lfloor d/2 \rfloor - 1}{2\lfloor d/2 \rfloor} n^2$.

Swanepoel, 2009: the exact value of $U_d(2,n)$ for even $d \geqslant 6$ and sufficiently large n and of $D_d(2,n)$ for $d \geqslant 4$ and sufficiently large n.

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Schur's conjecture

Conjecture, Schur et. al.

We have $D_d(d,n) = n$ for $n \geqslant d+1$.

They verified it for $d=3$. For $d=2$ it is a result due to H. Hopf, E. Pannvitz.

F. Morić and J. Pach proved that it is true provided that any two d-cliques in the diameter graph in $\mathbb{R}^d$ have at least $d-2$ vertices in common.

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The main theorem

Theorem 2 (AK, 2013)

1. For $n \geqslant 52$ we have

$$D_4(2,n) = \begin{cases} \lfloor n/2 \rfloor \lceil n/2 \rceil + \lceil n/2 \rceil + 1, & \text{if } n \not\equiv 3 \bmod 4, \\ \lfloor n/2 \rfloor \lceil n/2 \rceil + \lceil n/2 \rceil, & \text{if } n \equiv 3 \bmod 4 \end{cases}$$

(this part of the theorem in case of sufficiently large n is due to K. Swanepoel).

2. For all sufficiently large n we have

$$D_4(3,n) = \begin{cases} (n-1)^2/4 + n, & \text{if } n \equiv 1 \bmod 4, \\ (n-1)^2/4 + n - 1, & \text{if } n \equiv 3 \bmod 4, \\ n(n-2)/4 + n, & \text{if } n \equiv 0 \bmod 2. \end{cases}$$

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Ideas of the proof. Theorem 1, part 1.

- Decompose the graph into maximal complete bipartite subgraphs. Use Kővári-Sós-Turán theorem.
- Complete bipartite subgraphs lie on two orthogonal circles.
- There are "few" edges between the parts.
- By induction on the number of vertices show that in the extremal case there is only one (spanning) complete bipartite subgraph.
- Analyze the case with one spanning complete bipartite subgraph.

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- Additionally, we use Theorem 1 to estimate the number of triangles containing one vertex and the total number of triangles.
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We sketch the proof of the fact that any diameter graph G in $\mathbb{R}^3$ on n vertices has at most $2n-2$ edges. Moreover, any two odd cycles in G must share a vertex.

- Form the set of all directions of edges of a diameter graph. Each edge corresponds to two points on the sphere.
- Vertex v of $G \to$ two sets $R(v), B(v)$: diametrally opposite on the sphere, $R(v)$ is a spherical convex hull of the directions of edges going out from v.
- For distinct $u, v \in V(G)$ the sets $R(u), R(v)$ do not intersect. The same holds for $B(u), B(v)$. The sets $R(u), B(v)$ may share one vertex, this corresponds to an edge.
- Interpret the sets $R(u), B(v)$ as the vertices of a graph H and intersections between them as the edges. It is a planar bipartite graph, which is a double cover of the original graph.
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Open problems

- The proof of Schur's conjecture is the topic of Alexandr Polyanskiy's talk.
- Analogue of Theorem 1 for radii $\sqrt{3/8} < r < 1/\sqrt{2}$.
- Is it true that in a diameter graph in $\mathbb{R}^4$ any two 4-chromatic subgraphs must share an edge?
- How many edges there may be in a diameter graph in $\mathbb{R}^4$ apart from the edges in its maximal complete bipartite subgraph? Is this quantity linear in the number of vertices?
- Borsuk's conjecture for finite sets in $\mathbb{R}^4$?

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