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Explicit and probabilistic constructions of distance graphs with small clique numbers and large chromatic numbers

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Abstract. We study distance graphs with exponentially large chromatic numbers and without k-cliques, that is, complete subgraphs of size k. Explicit constructions of such graphs use vectors in the integer lattice. For a large class of graphs we find a sharp threshold for containing a k-clique. This enables us to improve the lower bounds for the maximum of the chromatic numbers of such graphs. We give a new probabilistic approach to the construction of distance graphs without k-cliques, and this yields better lower bounds for the maximum of the chromatic numbers for large k.

Keywords: distance graph, chromatic number, clique number, Nelson problem.

§ 1. Introduction

In this paper we study the chromatic numbers of distance graphs (see [1]) of a special kind. A graph $G=(V,E)$ is called a *distance graph* if V is a subset of $\mathbb{R}^n$ and

$$E \subseteq \bigl\{\{x,y\} \mid x,y \in V,\ |x-y|=a\bigr\}$$

for some fixed a.

Such graphs naturally arise in the problem of finding the chromatic number $\chi(\mathbb{R}^n)$ of the space $\mathbb{R}^n$. Fix a positive real number a. The *chromatic number* of $\mathbb{R}^n$ is the minimum number of colours required to colour the points of the space in such a way that no points of the same colour are at distance a from each other. The specific value of a is irrelevant (use a homothety of the sets coloured) and we put it equal to 1. Here is a formal definition of the chromatic number (see [2]):

$$\chi(\mathbb{R}^n)=\min\{m\in\mathbb{N} \mid \mathbb{R}^n=H_1\cup\dots\cup H_m \colon \forall\, i,\ \forall\, x,y\in H_i \quad |x-y|\neq 1\}.$$

The problem of finding this number for the plane was posed by Nelson in 1950 (see [3]). It turned out to be very difficult, and the value of the chromatic number is still unknown even in this case. However, a large number of related results have been obtained (see the surveys [4], [5]). We state some bounds for $\chi(\mathbb{R}^n)$ with small values of n:

$$4\leqslant\chi(\mathbb{R}^2)\leqslant 7, \qquad 6\leqslant\chi(\mathbb{R}^3)\leqslant 15$$

(the last estimate was obtained in [6], [7]) and, moreover,

$n=$	4	5	6	7	8	9	10	11	12
$\chi(\mathbb{R}^n)\geqslant$	7 [8]	9 [8]	11 [9]	15 [4]	16 [10]	21 [11]	23 [11]	25 [12]	27 [13]

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In what follows we study the asymptotic behaviour of $\chi(\mathbb{R}^n)$ as $n \to \infty$. It is known that

$$(\zeta_{\text{low}} + \bar{o}(1))^n \leqslant \chi(\mathbb{R}^n) \leqslant (3 + \bar{o}(1))^n, \qquad \text{where} \quad \zeta_{\text{low}} = 1.239\dots$$

(see [4], [10]).

By the Erdős–de Bruijn theorem (see, for example, [4]) the chromatic number of a space is equal to the chromatic number of a certain finite distance graph in this space. At the same time, Erdős [14] proved that, given k and l, one can construct an abstract graph with chromatic number greater than k containing no cliques of length less than l (in particular, no cliques of length 3 or less).

We wonder how the chromatic number of a distance graph behaves if we additionally require that the graph contains no cliques or cycles of a fixed size. Erdős [15] asked whether there is a planar distance graph with chromatic number 4 without triangles. This question was answered in the affirmative. Moreover, O'Donnell (see [16], [17]) proved that for every $l \in \mathbb{N}$ there is a planar distance graph with chromatic number 4 without cycles of length l or less.

We consider only the case of cliques. Let $\chi_k(\mathbb{R}^n)$ be the maximum of the chromatic numbers of the distance graphs in $\mathbb{R}^n$ containing no cliques of size k, where $k \geqslant 3$. We are interested in the behaviour of $\chi_k(\mathbb{R}^n)$. It is known (see, for example, [18]) that $\chi_k(\mathbb{R}^n) \geqslant (c_k + \bar{o}(1))^n$ for some constant $c_k > 1$. We define the quantity

$$\zeta_k = \sup\bigl\{c\colon \exists\, \delta(n) = \bar{o}(1),\ \forall\, n\ \exists \text{ a distance graph } G \text{ in } \mathbb{R}^n \\ \text{containing no cliques of size } k \text{ and such that } \chi(G) \geqslant (c + \delta(n))^n\bigr\}.$$

By what was said above, $\zeta_k > 1$. It was proved in [18] that $\zeta_k \geqslant c_k$, where c_k approaches ζ_{low} as k grows. The proof is probabilistic. The same paper contains explicit constructions of graphs with exponentially large chromatic numbers and without cliques of a given size. These constructions do not give an optimal constant for large k but do enable one to obtain more precise estimates for small k.

All the known explicit constructions of sequences of distance graphs with exponentially large chromatic numbers are of a specific form (see, for example, [19]).

Definition 1. The *family* (*class*) *of graphs* consists of the graphs $G = G(n, m, \{a_0, a_1, \dots, a_m\}, x) = (V, E)$ with vertex set

$$V = \biggl\{x = (x_1, \dots, x_n)\colon x_i \in \{0, 1, \dots, m\},\ |\{i\colon x_i = j\}| = a_j n\ \forall\, j = 0, \dots, m, \\ a_i \in (0, 1),\ \sum_{i=0}^{m} a_i = 1\biggr\}$$

and edge set $E = \bigl\{\{y_1, y_2\} \mid y_1, y_2 \in V,\ (y_1, y_2) = xn\bigr\}$. Here $(\cdot\,,\cdot)$ is the scalar product.

Remark 1. We shall often assume that n is divisible by a certain number. For example, in this construction we certainly assume that $a_j n$, xn are integers.

Remark 2. The probabilistic construction in [18] also uses graphs of a similar form, with some randomly chosen edges deleted.

In this paper we get an asymptotically (in n) exact bound for the presence of k-cliques in the graphs described above and, as a corollary, new bounds for ζ_k for small values of k. Our construction is explicit and optimal for our method.

We also sharpen the probabilistic result in [18] and thus improve the bounds for ζ_k for any k.

The structure of the paper is as follows. In § 2 we state our results and compare them with previously known facts. In particular, § 2.1 contains the results on the presence of cliques for $(0,1,\dots,m)$-vectors, and § 2.2 contains the results on the space chromatic numbers. The results in § 2.1 (resp. § 2.2) are proved in § 3 (resp. § 4).

§ 2. Statement of results and comparison with previous bounds

This section is divided into two parts. The first is devoted to results on the presence of cliques in the graphs described above, and in the second, we state the bounds for ζ_k resulting from the first part and compare them with previously known bounds. Moreover, § 2.2.2 contains the results on ζ_k obtained by the probabilistic method, and § 2.2.3 contains comparison tables of the bounds.

2.1. Cliques and $(0,1,\dots,m)$-vectors. We write $[x]$, $\{x\}$ and $\lceil x\rceil$ respectively for the integer and fractional parts of x and the integer closest to x from above. We first consider the case of $(0,1)$-graphs. Their definition is obtained from Definition 1 by putting $m=1$.

Theorem 1. *Suppose that $a,x>0$, $a,x\in\mathbb{Q}$. Consider a sequence of dimensions $n_1,n_2,\dots$ such that $an_i, xn_i\in\mathbb{N}$ for $i\in\mathbb{N}$, and the distance graphs $G_i=G(n_i,1,\{1-a,a\},x)$. Let k be an integer, $k\geqslant 3$. If*

$$x<\frac{(ka)^2-\{ka\}^2-[ka]}{k(k-1)}=:f_1, \tag{1}$$

then G_i contains no complete subgraphs (cliques) on k vertices. Moreover, this estimate is exact in the following sense. There is a constant $c=c(k,a)$ such that the graphs $\widetilde{G}_i=G(cn_i,1,\{1-a,a\},f_1)$, $i\in\mathbb{N}$, for the sequence of dimensions $cn_1,cn_2,\dots$ contain complete subgraphs on k vertices.

Remark 3. One can choose $\operatorname{LCM}\bigl(\binom{k}{[ka]},\binom{k}{\lceil ka\rceil}\bigr)$ for c. Then $c\mid\bigl(\binom{k}{\lceil ka\rceil}\lceil ka\rceil\bigr)$. In particular, c admits an upper bound independent of a because, for example, $c|k!$.

We now pass to the case of $(-1,0,1)$-graphs. This class is very similar to, but different from, the class of graphs $G(n,2,\{b,1-a-b,a\},x)$. Therefore we give an independent definition.

Definition 2. The *family* (*class*) *of graphs* consists of the graphs $G'(n,\{b,1-a-b,a\},x)=(V,E)$, where

$$V=\bigl\{x=(x_1,\dots,x_n)\colon x_i\in\{-1,0,1\},\ |\{i\colon x_i=1\}|=an,\ |\{i\colon x_i=-1\}|=bn,\ a,b\in(0,1)\bigr\},$$
$$E=\bigl\{\{y_1,y_2\}\mid y_1,y_2\in V,\ (y_1,y_2)=-xn\bigr\}.$$

Theorem 2. *Suppose that* $a, b, x > 0$, $a, b, x \in \mathbb{Q}$, $a \geqslant b$. *Consider a sequence of dimensions* $n_1, n_2, \dots$ *such that* $an_i, bn_i, xn_i \in \mathbb{N}$ *for* $i \in \mathbb{N}$, *and the graphs* $G_i = G'(n_i, \{b, 1-a-b, a\}, x)$. *Let* k *be an integer,* $k \geqslant 3$.

1) *Suppose that* $\{ka\} + k(1-a-b) \geqslant 1$. *If*

$$x > \frac{k(a+b) - (k(a-b))^2 - \{k(a-b)\} + \{k(a-b)\}^2}{k(k-1)} =: f_1, \tag{2}$$

then the graphs G_i *contain no* k*-cliques. The bound* (2) *is exact for some subsequence of dimensions. More precisely, there is a constant* $c \in \mathbb{N}$ *such that the graphs* $G'(cn_i) = G'(cn_i, \{b, 1-a-b, a\}, f_1)$, $i \in \mathbb{N}$, *for the sequence of dimensions* $cn_1, cn_2, \dots$ *contain cliques of size* k.

2) *Suppose that* $\{ka\} + k(1-a-b) < 1$. *If*

$$x > \frac{(k-k^2)(2a+2b-1) + (4k-4)\{ka\} + 4\{ka\}^2 + 4k(a+b)[ka] - 4(ka)^2}{k(k-1)} =: g_1, \tag{3}$$

then the graphs G_i *contain no* k*-cliques. Moreover, there is a constant* $c \in \mathbb{N}$ *such that the graphs* $G'(cn_i) = G'(cn_i, \{b, 1-a-b, a\}, g_1)$, $i \in \mathbb{N}$, *for the sequence of dimensions* $cn_1, cn_2, \dots$ *contain cliques of size* k.

Remark 4. In Theorem 2, the restrictions (2), (3) are lower bounds for x since the prohibited scalar product is equal to $-xn$. The desired constant c divides $k!$ in both cases. Moreover, the restriction (3) is stronger than (2), although this is not apparent from their formulations.

We now consider the general case.

Theorem 3. *Suppose that* $a_i, x > 0$, $a_i, x \in \mathbb{Q}$, *where* $i = 0, \dots, m$. *Consider a sequence of dimensions* $n_1, n_2, \dots$ *such that* $a_i n_j, xn_j \in \mathbb{N}$ *for all* $i = 0, \dots, m$ *and* $j \in \mathbb{N}$, *and the distance graphs* $G_i = G(n_i, m, \{a_0, a_1, \dots, a_m\}, x)$. *Take* $k \in \mathbb{N}$, $k \geqslant 3$. *If*

$$x < \frac{\left(k\sum_{i=0}^m ia_i\right)^2 - \left\{k\sum_{i=0}^m ia_i\right\}^2 + \left\{k\sum_{i=0}^m ia_i\right\} - k\sum_{j=0}^m j^2 a_j}{k(k-1)} =: f_1, \tag{4}$$

then the graphs G_i *contain no* k*-cliques.*

Under the same hypotheses suppose that $\delta \in \mathbb{Z}$, $\delta \leqslant k\sum_{i=0}^m ia_i < \delta + 1$, *and the system of equations*

$$\sum_{i=0}^m iq'_i \in \{\delta, \delta+1\}, \qquad \sum_{i=0}^m q'_i = k, \tag{5}$$

has an integer solution. Let $\mathbf{q}^j = (q_0^j, \dots, q_m^j)$ *be all non-negative integer solutions of the system* (5), *and let* u *be the* LCM *of the denominators of the coefficients in all possible representations of* $k\mathbf{a} = (ka_0, \dots, ka_m)$ *as a convex linear combination of the* $\mathbf{q}^j$. *Then the graphs* $G_i = G(k!\,un_i, m, \{a_0, a_1, \dots, a_m\}, f_1)$, $i \in \mathbb{N}$, *for the sequence of dimensions* $k!\,un_1, k!\,un_2, \dots$ *contain* k*-cliques.*

Remark 5. The system (5) actually describes the restrictions imposed on $\mathbf{a}$ (the vector $k\mathbf{a}$ must lie in the convex hull of solutions of the system). However, one can sometimes get rid of it. The second part of Theorem 3 has the following corollary. Under the same hypotheses suppose that $a_i \geqslant \frac{1}{k}$, $i = 1, \dots, m$. Then the graphs $G_i = G(k!\,n_i, m, \{a_0, a_1, \dots, a_m\}, f_1)$, $i \in \mathbb{N}$, for the sequence of dimensions $k!\,n_1, k!\,n_2, \dots$ contain k-cliques.

Remark 6. If we put $m = 1$, $a_1 = a$ in Theorem 3, then the estimate (4) coincides with the estimate (1) in Theorem 1. In the same vein, substituting $m = 2$, $a_0 = b$, $a_2 = a$ in Theorem 3, we obtain that the estimate (4) coincides with the estimate (2) in Theorem 2, although this is not immediately obvious (one must pass from $(-1, 0, 1)$-vectors to $(0, 1, 2)$-vectors).

Moreover, Theorem 3 does not cover the case corresponding to part 2 of Theorem 2. Therefore, in particular, Theorem 3 is not asymptotically sharp for all values of the a_i.

2.2. Chromatic numbers. Here we discuss the behaviour of the constant ζ_k for various values of k. The best bounds known so far were obtained in [18], where two approaches were suggested.

The first of these is expressed by Theorem 4 in [18]. It proceeds by explicitly constructing distance graphs without cliques. Namely, one uses the parameters a, x to optimize the chromatic number of $(0,1)$-graphs of the form $G(n, 1, \{1-a, a\}, x)$ containing no cliques of size k.

The second approach is probabilistic and is realized in Theorems 6, 7 in [18]. Consider the families of graphs $G(n, 1, \{1-a, a\}, x)$ and $G'(n, \{b, 1-a-b, a\}, x)$, which may initially contain cliques. To get a graph without cliques, one deletes some of their edges in a random way.

The graphs obtained in [18], Theorem 4, differ significantly from those obtained in Theorems 6, 7. This difference goes beyond the mere fact that the former is explicit.

All pairs of vertices at distance one are connected by edges in the first case but not in the second. Thus the definition of the set of edges of a distance graph is an equality in the first case and a strict inclusion in the second. The second approach yields a (non-induced) subgraph of $G(n, 1, \{1-a, a\}, x)$ or $G'(n, \{b, 1-a-b, a\}, x)$.

The results in this subsection are divided into two groups. § 2.2.1 contains strengthened and generalized versions of Theorem 4 in [18]. Theorems 1, 2 enable us to obtain stronger estimates for ζ_k, which are optimal in the framework of this method (this is clear from their statements). The two theorems in § 2.2.2 improve Theorems 6, 7 of [18].

The tables at the end of the section contain all the estimates, old and new, explicit and implicit.

The behaviour of ζ_k as $k \to \infty$ is of special interest. Theorem 4 in [18] yields the constant $\frac{3^{0.75}}{2} = 1.139\dots$ in the limit. A similar constant is given by an improvement (to be stated below) of this theorem for graphs of the form $G(n, 1, \{1-a, a\}, x)$. Extending this approach to the graphs $G'(n, \{b, 1-a-b, a\}, x)$, we get the constant $\frac{2}{\sqrt{3}} = 1.154\dots$ in the limit. All these constants are much smaller than the constant ζ_{low} whose powers occur in the best known estimate for the chromatic number of the space. But the probabilistic approach in [18] (as well as our approach in § 2.2.2) yields precisely this constant in the limit.

The following interesting question is still open.

Question 1. Can one *explicitly* construct a family of graphs without cliques of size k in such a way that the analogous constant (as $n \to \infty$) tends to ζ_{low} as k grows?

This can probably be answered using Theorem 3, but the difficulty of the optimization problem prevents us from making the corresponding calculations within the framework of this paper.

The following seemingly easier question arises from an analysis of the probabilistic construction.

Question 2. Is there a construction (explicit or not) to prove the existence of a family of graphs (depending on n and k) such that the resulting estimates of ζ_k tend to ζ_{low} as k grows, and the graphs have the *complete distance* property, that is, the definition of the set of edges holds with equality?

2.2.1. *Explicit constructions.*

Theorem 4. *Let x, a be such that $x > 0$ and $0 < a < 1$. Take an integer $k \geqslant 3$. If*

$$x < \frac{(ka)^2 - \{ka\}^2 - [ka]}{k(k-1)},$$

then we have

$$\zeta_k \geqslant \frac{(a-x)^{a-x}(1-a+x)^{1-a+x}}{a^a(1-a)^{1-a}}.$$

Theorem 5. *Let a, b, x be such that $a,b > 0$, $a \geqslant b$, $a+b < \frac{1}{2}$. Take an integer $k \geqslant 3$ and suppose that*

$$x > \frac{k(a+b) - (k(a-b))^2 - \{k(a-b)\} + \{k(a-b)\}^2}{k(k-1)}.$$

Put $p' = a+b+x$, $l = \frac{3p'+1-\sqrt{1+6p'-3(p')^2}}{6}$. Then

$$\zeta_k \geqslant \frac{l^l(p'-2l)^{p'-2l}(1-p'+l)^{1-p'+l}}{a^a b^b (1-a-b)^{1-a-b}}.$$

2.2.2. *Probabilistic constructions.* The method for proving the theorems in this subsection is similar to that of [18], but yields better estimates by using more powerful probabilistic tools and a more detailed analysis of the structure of the graph.

Before stating the results, we introduce some notation. Let $G(i)$ be a sequence of graphs on N_i vertices, $i \in \mathbb{N}$, $N_i \to \infty$ as $i \to \infty$. Let $\mathrm{conn}_k^j(G(i), v_1, \dots, v_j)$, $j < k$, be the number of k-cliques in $G(i)$ that contain the vertices $v_1, \dots, v_j$. We put

$$\mathrm{conn}_k^j(G(i)) = \max_{v_1,\dots,v_j \in G(i)} \mathrm{conn}_k^j(G(i), v_1, \dots, v_j). \tag{6}$$

In particular, $\mathrm{conn}_1^0(G_i) = N_i$, and $\mathrm{conn}_2^1(G_i)$ is the maximum degree of a vertex. It is easy to see that

$$\mathrm{conn}_{j_0}^{j_0-j}(G_i) \geqslant \mathrm{conn}_{j_1}^{j_1-j}(G_i) \quad \text{for} \quad j \leqslant j_0 \leqslant j_1.$$

We define a number $s_k^j(G(i))$ by the formula

$$s_k^j(G(i)) = \lim_{i\to\infty} \log_{N_i} \mathrm{conn}_k^j(G(i))$$

if the limit exists. Put $s_k^j(a,x) = s_k^j(G(n_i,1,\{1-a,a\},x))$ and $s_k^j(a,b,x) = s_k^j(G'(n_i,\{b,1-a-b,a\},x))$. We claim that the limit exists for these classes of graphs.

Indeed, for every j-tuple of vertices in any of the graphs $G(n_i,1,\{1-a,a\},x)$ or $G'(n_i,\{b,1-a-b,a\},x)$, the number of ways to complement these vertices to a k-clique is either equal to zero or equal to a sum of products of certain binomial coefficients. Therefore $\mathrm{conn}_k^j(G(i))$ can be written as $(\mathrm{const}(k,j)+\delta(n_i))^{n_i}$, where $\delta(n_i) \to 0$ as $i \to \infty$. The number of vertices in any of the graphs described is equal to $(\mathrm{const}+\delta'(n_i))^{n_i}$, where $\delta'(n_i) \to 0$ as $i \to \infty$. In this notation we have

$$\lim_{i\to\infty} \log_{N_i} \mathrm{conn}_k^j(G(i)) = \log_{\mathrm{const}} \mathrm{const}(k,j),$$

whence the limit with respect to i exists.

Theorem 6. *Take an arbitrary integer $k \geqslant 3$ and an arbitrary real number $a \in \left(0,\frac{1}{2}\right)$ and put*

$$\tau_0 = \tau_0(a) = \left(\frac{a}{2}\right)^{-\frac{a}{2}}\left(1-\frac{a}{2}\right)^{-(1-\frac{a}{2})}, \qquad \tau_1 = \tau_1(a) = a^{-a}(1-a)^{-(1-a)}.$$

Then $\zeta_k \geqslant c := \dfrac{\tau_1^{1-\frac{2s_k^2(a,a/2)}{(k-2)(k+1)}}}{\tau_0}$.

One can easily verify that τ_0 is always smaller than τ_1 under the hypotheses of Theorem 6. As a result, we have

$$\zeta_k \geqslant \max_{a\in(0,1/2)} \frac{(\tau_1(a))^{1-\frac{2s_k^2(a,a/2)}{(k-2)(k+1)}}}{\tau_0(a)}.$$

Theorem 7. *Take an arbitrary integer $k \geqslant 4$ and any real numbers b, a such that*

$$b,a \in (0,1), \qquad b+a \leqslant \frac{1}{2}, \qquad b \leqslant a,$$

and put

$$A = \frac{2+9b+3a-\sqrt{(2+9b+3a)^2-12(3b+a)^2}}{12},$$

$$B = \frac{3b+a}{2} - 2A, \qquad C = 1 + A - \frac{3b+a}{2}.$$

Furthermore, put

$$\rho_0 = \rho_0(a,b) = A^{-A}B^{-B}C^{-C}, \qquad \rho_1 = \rho_1(a,b) = b^{-b}a^{-a}(1-b-a)^{-(1-b-a)}.$$

Then $\zeta_k \geqslant c := \dfrac{\rho_1^{1-\frac{2s_k^2(a,b,(a-b)/2)}{(k-2)(k+1)}}}{\rho_0}$.

One can verify that $\rho_0 < \rho_1$ under the hypotheses of Theorem 7 (a proof can be found, for example, in [19]). As a result, we have

$$\chi_k(\mathbb{R}^n) \geqslant \max_{a,b} \frac{(\rho_1(a,b))^{1-\frac{2s_k^2(a,b,(a-b)/2)}{(k-2)(k+1)}}}{\rho_0(a,b)}.$$

However, to apply Theorem 6 or Theorem 7, we must have an upper bound for the coefficients $s_k^2(a,\frac{a}{2})$, $s_k^2(a,b,\frac{a-b}{2})$.

Table 1

k	Estimates in [18]	New estimate using Theorem 4	New estimate using Theorem 5
	$\zeta_k \geqslant$		
3	1.0582	1.0582	–
4	1.0582	**1.0663**	1.0374
5	1.0582	**1.0857**	1.0601
6	1.0743	**1.0898**	1.0754
7	1.0857	**1.0995**	1.0865
8	1.0933	**1.1019**	1.0948
9	1.0992	**1.1077**	1.1013
10	1.1033	**1.1093**	1.1066
11	1.1075	**1.1131**	1.1109
12	1.1096	1.1142	**1.1145**
13	1.1124	1.1170	**1.1175**
14	1.1151	1.1178	**1.1201**
15	1.1220	1.1198	**1.1224**
$\lim_{k\to\infty}$	**1.239**	1.139	1.154

Proposition 1. *For all i,j,k with $k \geqslant i \geqslant j$ and any sequence of graphs $G(l)$ whose number of vertices tends to infinity as $l \to \infty$, we have*

$$s_k^j(G(l)) \leqslant s_i^j(G(l)) + (k-i)s_{i+1}^i(G(l)).$$

Proof. This follows easily since $s_{i+c}^i(G(l))$ is a decreasing function of i for every positive integer c and $s_{i+c}^i(G(l)) \leqslant s_{i+c_0}^i(G(l)) + s_{i+c}^{i+c_0}(G(l))$ for all $i, c_0, c \in \mathbb{N}$, $c_0 \leqslant c$. □

To get sufficiently precise estimates for $s_k^2(a,\frac{a}{2})$ and $s_k^2(a,b,\frac{a-b}{2})$, one must solve difficult optimization problems. We do not dwell on this but rather use a simple estimate based on Proposition 1.

Table 2

k	Theorem 6 [18]	Theorem 6 and Prop. 2	Theorem 7 [18]	Theorem 7
	$\zeta_k \geqslant$			
3	–	**1.0147**	–	–
4	–	**1.0321**	–	1.0028
5	1.0029	**1.0491**	1.0028	1.0169
6	1.0183	**1.0641**	1.0169	1.0339
7	1.0362	**1.0771**	1.0339	1.0501
8	1.0524	**1.0881**	1.0501	1.0646
9	1.0663	**1.0976**	1.0646	1.0773
10	1.0781	**1.1057**	1.0773	1.0886
11	1.0882	**1.1128**	1.0886	1.0985
12	1.0970	**1.1190**	1.0985	1.1073
13	1.1045	**1.1245**	1.1073	1.1151
14	1.1112	**1.1293**	1.1151	1.1220
15	1.1170	**1.1336**	1.1220	1.1283
16	1.1222	**1.1375**	1.1283	1.1339
17	1.1268	**1.1409**	1.1339	1.1390
18	1.1309	**1.1441**	1.1390	1.1437
19	1.1347	1.1470	1.1437	**1.1479**
20	1.1381	1.1496	1.1479	**1.1518**
100	1.1926	1.1943	1.2195	**1.2197**
1000	1.2056	1.2058	1.2375	**1.2375**
10000	1.2069	1.2069	1.2393	**1.2393**
100000	1.2070	1.2070	1.2395	**1.2395**
1000000	1.2071	1.2071	1.2395	**1.2395**

Proposition 2. *The following estimates hold*:

$$s_k^2\left(a, b, \frac{a-b}{2}\right) \leqslant k-2, \tag{7}$$

$$\begin{aligned} s_k^2\left(a, \frac{a}{2}\right) &\leqslant (k-2)s_3^2\left(a, \frac{a}{2}\right) \\ &= (k-2)\log_{\frac{1}{a^a(1-a)^{1-a}}}\left(\frac{\left(1-\frac{3a}{2}\right)^{1-\frac{3a}{2}}\left(\frac{a}{2}\right)^{\frac{3a}{2}}}{t^{4t}\left(\frac{a}{2}-t\right)^{\frac{3a}{2}-3t}\left(1-\frac{3a}{2}-t\right)^{1-\frac{3a}{2}-t}}\right), \end{aligned} \tag{8}$$

where t is a root of the equation $t^4 = \left(\frac{1}{2}a - t\right)^3\left(1 - \frac{3}{2}a - t\right)$. More precisely,

$$t = \left(\frac{1}{12}\sqrt[3]{216\,a^2 - 54\,a - 216\,a^3 + 6\,\sqrt{-264\,a^3 + 216\,a^4 + 81\,a^2}} - 12\,\frac{\frac{1}{6}\,a - \frac{1}{4}\,a^2}{\sqrt[3]{216\,a^2 - 54\,a - 216\,a^3 + 6\,\sqrt{-264\,a^3 + 216\,a^4 + 81\,a^2}}} - \frac{1}{2}\,a + \frac{1}{2}\right)a. \tag{9}$$

Remark 7. The numerical bounds for ζ_k obtained from Theorem 6 use the estimate (8). To get the definitive value, one performs optimization with respect to a.

Unfortunately, we did not get a precise estimate even for $s_2^1(a, b, \frac{a-b}{2})$ because this would require solving systems of equations of large degree, which is very difficult computationally. Therefore we have used a trivial estimate in the $(-1, 0, 1)$-case.

2.2.3. *Comparison tables of estimates.* Finally, we give the numerical values of the lower bounds for ζ_k obtained by optimizating the expressions in Theorems 4, 5 with respect to the parameters a, b, x, and compare them with previously known bounds. We again stress that the new bounds in Table 1 (where the best of the three constants are boldfaced) are obtained using explicit constructions.

In Table 2 we state the bounds following from our Theorems 6, 7 and compare them with the implicit bounds in Theorems 6, 7 of [18].

Remark 8. Table 2 (and the statement of Theorem 7) show that if we use the trivial estimate $s_k^2(a, b, \frac{a-b}{2}) \leqslant k-2$, then our bounds for ζ_k are the bounds in [18] shifted by 1 (the bounds valid for $k = i+1$ now hold for $k = i$).

§ 3. Proofs. Cliques and $(0, 1, \dots, m)$-vectors

Remark 9. It is known that the radius of a sphere circumscribed around a simplex with sides of unit length and k vertices is equal to $\sqrt{\frac{k-1}{2k}}$. Hence, if a graph contains a k-clique as a subgraph, then its realization as a distance graph with edges of unit length cannot be inscribed in a sphere whose radius is smaller than the quantity indicated above.

This simple observation actually yields good estimates in the problem under consideration. However we look for even more precise results.

3.1. The construction with $\{0, 1\}$-vectors. The first class of distance graphs to be analyzed is $G(n, 1, \{1-a, a\}, x) = (V, E)$. In other words,

$$V = \bigl\{x = (x_1, \dots, x_n) \colon x_i \in \{0, 1\},\ x_1 + \dots + x_n = an,\ a \in (0, 1)\bigr\},$$

and vertices x, y are connected by an edge if and only if $(x, y) = xn$, where $x \in (0, a)$.

We first calculate the radius of a sphere containing V under the normalization that makes the prohibited distance equal to one. The centre of this sphere lies in the plane $\sum_{i=1}^n x_i = an$ or, more precisely, at the point $(a, \dots, a)$. Denoting the radius of the sphere by r', we have

$$(r')^2 = (1-a)^2 an + a^2(n-an) = na(1-a). \tag{10}$$

Then the length l of every edge satisfies $l^2 = (2a-2x)n$ by the cos rule. If we normalize our set to make the diameter equal to one, then the radius r of the normalized sphere satisfies

$$r^2 = \frac{(r')^2}{l^2} = \frac{a(1-a)}{2a-2x}.$$

In what follows we fix the parameter a and ask whether the graph contains k-cliques for various values of x.

3.1.1. *A restriction on x depending on k.* Suppose that the graph contains a k-clique. The k column-vectors that form the clique can be written as an $(n\times k)$ matrix. Let d_i be the *degree of the coordinate* i, that is, the number of vectors whose ith coordinate is equal to one (equivalently, the sum of the entries in the ith column of the matrix). We easily see that $\sum_{i=1}^n d_i = kan$: these are two expressions for the number of unit coordinates of the vectors forming the k-clique.

There is another equation expressing two ways to compute the total scalar product over all pairs of vectors. On one hand, it is equal to $\frac{k(k-1)}{2}xn$. On the other, each coordinate of degree d_i contributes $\frac{d_i(d_i-1)}{2}$ to the total scalar product. We get a system of two equations

$$\sum_{i=1}^n d_i = kan, \qquad \sum_{i=1}^n d_i(d_i-1) = k(k-1)xn. \tag{11}$$

Since a is fixed, the value of $\sum_{i=1}^n d_i$ is also fixed. We must find all possible values of x or, equivalently, all possible values of $\sum_{i=1}^n d_i(d_i-1)$.

We claim that the minimum of $\sum_{i=1}^n d_i(d_i-1)$ for a fixed sum $\sum_{i=1}^n d_i$ is attained if $|d_i-d_j|\leqslant 1$ for all i, j.

We first note that $\sum_{i=1}^n d_i(d_i-1) = \sum_{i=1}^n d_i^2 - kan$. Hence we can minimize $\sum_{i=1}^n d_i^2$. Suppose that there are i and j with $d_i - d_j = 2h+b$ with $h,b>0$. We claim that

$$d_i^2 + d_j^2 > (d_i-h)^2 + (d_j+h)^2 \tag{12}$$

and thus establish that the left-hand side of (12) takes its minimum value (for a fixed sum of the two degrees of coordinates) when these degrees are closest to each other. Indeed, rewrite (12) in the form

$$d_i^2 + d_j^2 > (d_i-h)^2 + (d_j+h)^2 \ \Leftrightarrow\ 0 > 2h^2 - 2h(d_i-d_j) = -2h^2 - 2hb.$$

We see that the inequality on the right always holds for $h,b>0$. Wishing to minimize x for a fixed a, we must have $|d_i-d_j|\leqslant 1$ for all i, j. Hence the degrees must be of the form either $d=[ka]$ or $d'=\lceil ka\rceil = d+1$. We easily see that the number of coordinates of degree d' is equal to $\{ka\}n$. Then the number of coordinates of degree d is equal to $(1-\{ka\})n$ and the system of equations takes the form

$$\begin{cases} dn + \{ka\}n = kan, \\ d^2(1-\{ka\})n + (d+1)^2\{ka\}n - kan = k(k-1)xn \end{cases}$$
$$\Leftrightarrow \begin{cases} d = [ka], \\ d^2 + (2d+1)\{ka\} - ka = k(k-1)x \end{cases}$$
$$\Leftrightarrow \begin{cases} d = [ka], \\ ([ka])^2 + (2[ka]+1)\{ka\} - ka = k(k-1)x. \end{cases}$$

It follows from the last equation of the system on the right that

$$x = \frac{([ka]+\{ka\})^2 + \{ka\} - \{ka\}^2 - ka}{k(k-1)} = \frac{(ka)^2 - \{ka\}^2 - [ka]}{k(k-1)} = f_1.$$

Hence, if we want the graph to contain a clique of size k, we must necessarily have $x \geqslant f_1$. In other words, if $x < f_1$, then the graph contains no cliques of size k.

3.1.2. *A simpler restriction on x and a comparison with the restriction in* § 3.1.1. We note that wishing to minimize x for a fixed a, we must have $d_1 = \dots = d_n = d$. Then the system of equations (11) is transformed into

$$\begin{cases} d = ka, \\ d(d-1) = k(k-1)x \end{cases} \Leftrightarrow \begin{cases} d = ka, \\ ka(ka-1) = k(k-1)x. \end{cases}$$

The last equation of the system on the right implies that $x = \frac{a(ka-1)}{k-1} =: f_2$. Thus if the graph contains a clique of size k, then $x \geqslant f_2$. Substituting $x = f_2$ in (10), we have

$$r^2 = \frac{a(1-a)}{2a - 2\frac{a(ka-1)}{k-1}} = \frac{(k-1)(1-a)}{2k-2ka} = \frac{k-1}{2k}.$$

The right-hand side is equal to the squared radius of the sphere circumscribed around a simplex on k vertices. Hence the restriction $x \leqslant f_2$ can be obtained in two ways, the first of which has already been described above and the second is geometric (see Remark 9). We realize the graph on a sphere in such a way that the vertices connected by an edge are at unit distance from each other. Then for the graph to contain a clique of size k, it is necessary that the radius r of the sphere containing the graph is greater than or equal to $\sqrt{\frac{k-1}{2k}}$.

In any case we obtain that the graph contains no cliques of size k if $x < f_2$. To compare the functions f_1 and f_2, we write

$$f_1 - f_2 = \frac{(ka)^2 - \{ka\}^2 - [ka]}{k(k-1)} - \frac{a(ka-1)}{k-1} = \frac{ka - \{ka\}^2 - [ka]}{k(k-1)} = \frac{\{ka\} - \{ka\}^2}{k(k-1)}. \tag{13}$$

Thus the bound (1) is sharper, but the asymptotics of (1) and (13) as $k \to \infty$ is the same and can be written in the form

$$\frac{a(1-a)}{2a-2x} = r^2 < \frac{1}{2}.$$

The estimate f_1 has the important advantage of being asymptotically exact (with respect to n for a fixed k). This will be shown in the next subsection.

3.1.3. *The construction.* Here we construct a k-clique in a graph with parameters a and $x = \frac{(ka)^2-\{ka\}^2-[ka]}{k(k-1)} = f_1$ such that the degree of $\{ka\}n$ coordinates is $d' = \lceil ka \rceil$, and the degree of the remaining $(1-\{ka\})n$ coordinates is $d = [ka]$. This proves that the bound (1) is asymptotically unimprovable.

Recall that an is assumed to be an integer, whence so is $\{ka\}n$. This holds for some sequence of dimensions $n_1, n_2, \dots$. Consider the sequence of dimensions $cn_1, cn_2, \dots$, where $c = \mathrm{LCM}\bigl(\binom{k}{d}, \binom{k}{d'}\bigr)$. We easily see that $c \mid \bigl(\binom{k}{\lceil ka \rceil}\lceil ka \rceil\bigr)$. Clearly, we have $\binom{k}{d} \mid (1-\{ka\})n$ and $\binom{k}{d'} \mid \{ka\}n$ for such a sequence.

We now consider the first $(1-\{ka\})n$ coordinates. Consider the block of the first $\binom{k}{d}$ coordinates and assume that k vectors forming a clique have the following structure on this block. The columns, each of which corresponds to a coordinate, give a list (without repetitions) of all possible ways to place d ones in k positions. It is easy to see that each vector contains the same number of ones in these positions (namely, $\binom{k-1}{d-1}$), and the scalar product of any two vectors (or the number of positions occupied by ones in both vectors) is the same for all pairs and is equal to $\binom{k-2}{d-2}$. Make copies of such blocks for all the $(1-\{ka\})n$ coordinates. The degree of each coordinate will be equal to d.

We do the same with the remaining $\{ka\}n$ coordinates: take their first $\binom{k}{d'}$ positions and require that k vectors forming a clique have the following structure on them. The columns, each of which corresponds to a coordinate, give a list (without repetitions) of all possible ways to place d' ones in k positions. Then we make copies of such blocks for all the $\{ka\}n$ coordinates.

Combining the results in this subsection, we get Theorem 1 as stated in § 2.1.

Remark 10. All the results of this subsection can be stated in terms of Hamming distances. Moreover, most of these results were known in these terms (see, for example, [20]).

3.1.4. *The case of irregular simplices.* The problem studied in this subsection can be generalized as follows. Suppose we are given a system of $(0,1)$-vectors with a fixed number of ones and a simplex on k vertices such that the scalar product of the vertices i, j is equal to x_{ij} (clearly, $x_{ij} = x_{ji}$). Under what assumptions on the x_{ij} can we guarantee that such a simplex is not contained in our system? Of course, we can guarantee this when the scalar product of some of the vectors turns out to be non-integer for such lengths of sides. One can also estimate x_{ij} as when obtaining the bound $x < f_1$. Modifying the system of equations (11), we get

$$\sum_{i=1}^{n} d_i = kan, \qquad \sum_{i=1}^{n} d_i(d_i-1) = \sum_{i,j,i\neq j} x_{ij}n.$$

Arguing as above, we conclude that if

$$\sum_{i,j,i\neq j} x_{ij} < (ka)^2 - \{ka\}^2 - [ka] = f_1',$$

then the system of $(0,1)$-vectors contains no simplices of the form described above.

3.2. The construction with $\{-1,0,1\}$-vectors. Here we consider the next class of graphs: the $(-1,0,1)$-graphs $G'(n,\{b,1-a-b,a\},x)=(V,E)$. For convenience, we recall from Definition 1 that

$$V=\bigl\{x=(x_1,\dots,x_n)\colon x_i\in\{-1,0,1\},\ |\{i\colon x_i=1\}|=an,\ |\{i\colon x_i=-1\}|=bn,\ a,b\in(0,1)\bigr\},$$
$$E=\bigl\{\{y_1,y_2\}\mid y_1,y_2\in V,\ (y_1,y_2)=-xn\bigr\}.$$

We first normalize the set V to make the edge-lengths equal to one. To compute the radius r' of the sphere containing this set, we note that the centre of this sphere is at the point $(a-b,\dots,a-b)$. Then

$$\begin{aligned}(r')^2&=an(1-a+b)^2+bn(-1-a+b)^2+(1-a-b)n(a-b)^2\\&=an(1-a+b-a+b)(1-a+b+a-b)\\&\qquad+bn(1+a-b-a+b)(1+a-b+a-b)+(a-b)^2n\\&=n(a(1-2a+2b)+b(1+2a-2b)+a^2-2ab+b^2)\\&=n(a+b-a^2-b^2+2ab)=n(a+b-(a-b)^2).\end{aligned}$$

Here the length l of an edge in the graph is given by the formula $l^2=2|v|^2-2(u,v)$, where u, v are the vectors in V forming the edge. We see that $l^2=2n(a+b)+2xn=2n(a+b+x)$. Normalizing the set V, we see that the radius r of the sphere satisfies

$$r^2=\frac{a+b-(a-b)^2}{2(a+b+x)}. \tag{14}$$

3.2.1. *The choice of parameters. A restriction on x.* As in § 3.1, we seek a restriction on x for fixed values of a, b, k. There is no loss of generality in assuming that $a\geqslant b$. Arguing as in § 3.1.1, we first analyze the contribution of a coordinate with a fixed number of plus ones and minus ones to the total scalar product.

Lemma 1. *Consider k vectors. Suppose that some coordinate of these vectors exhibits l minus ones and $l+\delta$ ones. Then the contribution d of this coordinate to the total product of these k vectors is equal to $-l+\frac{\delta^2-\delta}{2}$.*

Proof. Put $m=l+\delta$. It is easy to see that

$$d=\binom{l}{2}+\binom{m}{2}-ml=\frac{m(m-1)}{2}+\frac{l(l-1)}{2}-ml=\frac{1}{2}\bigl((m-l)^2-m-l\bigr). \tag{15}$$

Substituting $m=l+\delta$ in (15), we get $d=-l+\frac{\delta^2-\delta}{2}$. □

We now construct a system analogous to (11). Take k vectors that form a clique in a graph with parameters a, b, x. Let δ_i be the difference between the numbers of ones and minus ones occurring at the ith coordinate. Using two ways to calculate the difference between the numbers of ones and minus ones in the clique, we obtain that $\sum_{i=1}^n\delta_i=k(a-b)n$.

Let us calculate the total scalar product. On one hand, it is clearly equal to $-\binom{k}{2}xn$. On the other hand, Lemma 1 yields that the contribution of each coordinate is $d_i = -l_i + \frac{\delta_i^2-\delta_i}{2}$. We also know that the total number of minus ones is equal to kbn. Hence $\sum_{i=1}^n l_i = kbn$. We finally get the equations

$$\sum_{i=1}^n d_i = -kbn + \sum_{i=1}^n \frac{\delta_i^2 - \delta_i}{2} = -\binom{k}{2}xn.$$

Combining our results, we get a system

$$\begin{cases} \displaystyle\sum_{i=1}^n \delta_i = k(a-b)n, \\ \displaystyle k(k-1)xn = 2kbn - \sum_{i=1}^n \delta_i(\delta_i - 1) \end{cases} \Leftrightarrow \begin{cases} \displaystyle\sum_{i=1}^n \delta_i = k(a-b)n, \\ \displaystyle k(k-1)xn = k(a+b)n - \sum_{i=1}^n \delta_i^2. \end{cases} \tag{16}$$

We look for restrictions (as exact as possible) on x under which the graph contains no cliques of size k. This means that the sum of the numbers δ_i is fixed, and we choose them to minimize the sum of their squares and thus find the maximum value of x still allowing for the existence of a clique. Then there are certainly no cliques for larger x. It is easy to verify (as in § 3.1.1) that the minimum is attained in the case when the values of $\delta_1, \delta_2, \ldots, \delta_n$ are closest to each other. Let us see by how much they can differ.

Since the average contribution of each coordinate is equal to $k(a-b)$, in the very best case we obtain, as in § 3.1.1, $(1 - \{k(a-b)\})n$ coordinates of degree $\delta = [k(a-b)]$ and $\{k(a-b)\}n$ coordinates of degree $\delta' = 1 + \delta$.

Adding one to each coordinate of every vector, we increase the degree of each coordinate by k (we pass from δ to $d = \delta + k$ and from $(-1, 0, 1)$-vectors to $(0, 1, 2)$-vectors). Suppose that we have $\{ka\}n + k(1-a-b)n < n$. In other words, the sum of the total number of ones over all vectors in the clique (in the new set) and the remainder obtained when the total number of twos is divided into n is less than n. Then it can easily be seen that in the best case (for minimizing the sum of the squares of the degrees) there are $\{ka\}n$ coordinates of degree $d'' = d + 2$, $k(1-a-b)n$ of degree $d' = d + 1$ and $\bigl(1 - \{ka\}n - k(1-a-b)\bigr)n$ of degree d.

To justify these assertions, we first place twos in various coordinates. To minimize the differences between the degrees of the coordinates after the arrangement of twos, it would be optimal to have precisely $\{ka\}n$ coordinates of degree $d_2' = d_2 + 2$ and have all other coordinates of degree d_2. Then it is clear from the idea described in § 3.1.1 (minimization requires the degrees of the coordinates to be as close as possible) that the best way to place the ones in various coordinates is to increase the degrees of some coordinates from d_2 to $d_2 + 1$. Then it remains only to calculate the number of coordinates of each degree.

We can now return to $(-1, 0, 1)$-vectors and state the resulting restrictions on x. We first consider a very simple case when the coordinates have only two degrees.

Then the system (16) takes the form

$$\begin{cases}\delta n+\{k(a-b)\}n=k(a-b)n,\\ k(k-1)xn=k(a+b)n-\{k(a-b)\}n(\delta+1)^2-(1-\{k(a-b)\})n\delta^2\end{cases}$$

$$\Leftrightarrow\begin{cases}\delta=[k(a-b)],\\ k(k-1)x=k(a+b)-\delta^2-\{k(a-b)\}(2\delta+1)\end{cases}$$

$$\Leftrightarrow\begin{cases}\delta=[k(a-b)],\\ k(k-1)x=k(a+b)-\bigl([k(a-b)]+\{k(a-b)\}\bigr)^2-\{k(a-b)\}+\{k(a-b)\}^2\end{cases}$$

$$\Leftrightarrow\begin{cases}\delta=[k(a-b)],\\ x=\dfrac{k(a+b)-(k(a-b))^2-\{k(a-b)\}+\{k(a-b)\}^2}{k(k-1)}=:f_1.\end{cases}$$

It follows that the graph cannot contain cliques of size k if x is greater than f_1.

We now consider the second case, which is technically more difficult. Suppose that $\{ka\}n+k(1-a-b)n<n$. Then we already know that there are $\{ka\}n$ coordinates of degree $\delta''=d+2$, $k(1-a-b)n$ of degree $\delta'=\delta+1$, and the remaining $\bigl(1-\{ka\}n-k(1-a-b)\bigr)n$ have degree δ. We get a system

$$\begin{cases}\delta n+2\{ka\}n+k(1-a-b)n=k(a-b)n,\\ k(k-1)xn=k(a+b)n-\{ka\}n(\delta+2)^2-k(1-a-b)n(\delta+1)^2\\ \qquad\qquad -\bigl(1-\{ka\}n-k(1-a-b)\bigr)n\delta^2\end{cases}$$

$$\Leftrightarrow\begin{cases}\delta=k(2a-1)-2\{ka\}\ (=2[ka]-k),\\ k(k-1)x=k(a+b)-\{ka\}(4\delta+4)-k(1-a-b)(2\delta+1)-\delta^2=:g.\end{cases}$$

Rewrite the right-hand side of the last equation as

$$\begin{aligned} g&=k(2a+2b-1)-\{ka\}(8[ka]-4k+4)-k(1-a-b)(4[ka]-2k)-(2[ka]-k)^2\\ &=k(2a+2b-1)-\{ka\}(8[ka]-4k+4)+k(a+b)(4[ka]-2k)-(4[ka]^2-k^2)\\ &=k(2a+2b-1)+(4k-4)\{ka\}+4\{ka\}^2+4k(a+b)[ka]-k^2(2a+2b-1)\\ &\qquad -4([ka]+\{ka\})^2\\ &=(k-k^2)(2a+2b-1)+(4k-4)\{ka\}+4\{ka\}^2+4k(a+b)[ka]-4(ka)^2.\end{aligned}$$

Thus one can say that the graph contains no k-cliques under these restrictions provided that $x>\frac{g}{k(k-1)}$. This proves part 1 of Theorem 2.

Remark 11. Case 2 of Theorem 2 is not very interesting for our applications, but we need it to obtain an asymptotically exact estimate for all values of a, b, k.

Remark 12. As in § 3.1.4, case 1 of Theorem 2 can be stated for an arbitrary simplex.

In the next subsection we get a simplified but weaker bound. Then we prove that the bound in this subsection is exact (in the same sense as in § 3.1.3).

3.2.2. *A simpler restriction on x and a comparison with the previous one.* We could simplify things by acting as in § 3.1.2. Wishing to minimize x for fixed a, b, k, we must have $\delta_1 = \cdots = \delta_n = \delta$. Then the system of equations (16) takes the form

$$\begin{cases} \delta = k(a-b), \\ k(k-1)x = k(a+b) - \delta^2 \end{cases} \Leftrightarrow \begin{cases} \delta = k(a-b), \\ x = \dfrac{a+b}{k-1} - \dfrac{k(a-b)^2}{k-1} =: f_2. \end{cases}$$

Substituting $x = f_2$ in (14), we get

$$\begin{aligned} r^2 &= \frac{a+b-(a-b)^2}{2(a+b+x)} = \frac{a+b-(a-b)^2}{2\bigl(a+b+\frac{a+b}{k-1} - \frac{k(a-b)^2}{k-1}\bigr)} \\ &= \frac{a+b-(a-b)^2}{\frac{2k}{k-1}(a+b-(a-b)^2)} = \frac{k-1}{2k}. \end{aligned}$$

Here the right-hand side is again equal to the squared radius of the sphere circumscribed around the simplex on k vertices. Thus there are two ways to arrive at the restriction $x \geqslant f_2$. The first has been described above, and the second is geometric.

In any case, we get the following restriction: a graph $G'(n, \{b, 1-a-b, a\}, x)$ contains no cliques of size k if $x > f_2$. Comparing the functions f_1, g_1 and f_2, we get more or less the same result as in the $(0,1)$-case. All the bounds take the same asymptotic form with respect to k:

$$\frac{a+b-(a-b)^2}{2(a+b+x)} = r^2 < \frac{1}{2}.$$

However the bounds f_1, g_1 have an important advantage: they are asymptotically exact with respect to n for a fixed k. This will be proved in the next subsection.

3.2.3. *The construction.* We construct k-cliques in the graphs $G'(n, \{b, 1-a-b, a\}, f_1)$, $G'(n, \{b, 1-a-b, a\}, g_1)$. Consider cases 1, 2 of Theorem 2.

The case when $\{ka\}n + k(1-a-b)n \geqslant n$. This is the basic case. This restriction holds, for example, if the number of zeros is greater than n/k. Thus we must construct k vectors with a given number of ones and minus ones such that their scalar products are pairwise equal and the degrees of their coordinates differ by at most 1. In other words, there must be $(1 - \{k(a-b)\})n$ coordinates of degree $\delta = [k(a-b)]$ and $\{k(a-b)\}n$ of degree $\delta' = 1 + \delta$.

Recall that an, bn are assumed to be integers, whence so is $\{k(a-b)\}n$. This holds for some sequence of dimensions $n_1, n_2, \ldots$. Consider the sequence of dimensions $cn_1, cn_2, \ldots$, where c is a constant to be chosen later.

We start with an auxiliary construction in the dimensions n_i: construct a system of vectors with the prescribed total proportion of coordinates of each kind in such a way that the degrees of all coordinates differ at most by 1. Here we do not care about the form of these vectors or their scalar products. Our next step will be to symmetrize this construction (in a certain sense) by passing from the dimension n_i to cn_i, and thus obtain the desired construction.

For simplicity we write n instead of n_i. Hence the number of coordinates of the form $-1, 0, 1$ is equal to kbn, $k(1-a-b)n$, kan respectively. The first $[ka]$ vectors will consist only of ones. The next $[kb]$ vectors consist only of minus ones. We are left with $(\{ka\}+\{kb\}+k(1-a-b))n$ coordinates, where the coefficient of n is certainly an integer. By assumption, $(\{ka\}+k(1-a-b))n \geqslant n$. Hence one of the following two cases holds:

1) $(\{ka\}+k(1-a-b))n = n$, $\{kb\}=0$;

2) $(\{ka\}+\{kb\}+k(1-a-b)) \geqslant 2$.

Consider case 1). We are left with one vector to be filled with zeros and ones. Clearly, the degrees of the coordinates will differ by at most 1 for any such filling. The coordinate positions will be of two types: those containing $[ka]$ ones, $[kb]$ minus ones and one zero, and those containing $[ka]+1$ ones and $[kb]$ minus ones.

In case 2) we proceed as follows. We form pairs consisting of ones and minus ones (the number of such pairs is equal to $\min\{\{ka\}n, \{kb\}n\} < n$), place these pairs in two of the remaining (at least two) vectors, and then form one of these vectors using the remaining ones and minus ones and fill all its other coordinates with zeros. This yields columns of three types: those with $[ka]$ ones, $[kb]$ minus ones, and zeros otherwise; those with $[ka]+1$ ones, $[kb]+1$ minus ones, and zeros otherwise; and, finally, those with $[ka]$ ones, $[kb]$ minus ones, another one or minus one, and zeros otherwise. The degrees of the coordinates still differ by at most 1.

To symmetrize, we replace each coordinate by several blocks of coordinates as follows. Consider all possible permutations of the entries inside a column which give rise to different configurations. Their number is equal to $\prod_i \binom{k}{y_i}$ (a product of binomial coefficients, or a polynomial coefficient), where the y_i depend on the type of the column. The resulting set of columns is called a *block* of columns of the given type. We define c (the total number of columns that replace every one of the coordinate columns) to be the LCM of the sizes of the blocks for all possible types of columns. This guarantees that every block occurs an integer number of times among the coordinates of the vector. We easily see that $c|k!$.

Clearly, this construction preserves the proportions between the total numbers of coordinates, and it remains true that the degrees of the coordinates differ by at most 1. After symmetrization we obtain that the number of coordinates of any given type is the same for all vectors. The scalar products of any two vectors in the clique are also the same. Thus we have achieved the desired result by replacing the dimension n by cn, where c is a certain divisor of $k!$.

The case when $\{ka\}n + k(1-a-b)n < n$. We construct k vectors with given parameters and pairwise equal scalar products in such a way that the degrees of the coordinates differ by at most 2. More precisely, there will be $\{ka\}n$ coordinates of degree $\delta'' = d+2$, $k(1-a-b)n$ of degree $\delta' = \delta+1$, and the remaining $\bigl(1-\{ka\}n-k(1-a-b)\bigr)n$ have degree δ.

Then we argue as in the previous case, except for another auxiliary construction, which is as follows. The number of coordinates of the form $-1,0,1$ is equal to kbn, $k(1-a-b)n$, kan respectively. The first $[ka]$ vectors consist only of ones. The next vector consists of the remaining ones, all zeros (whose number is $\{ka\}n + k(1-a-b)n < n$), and some minus ones. The remaining vectors consist of minus ones only. All we need to verify before symmetrizing is that the degrees of the coordinates are as required. Then we perform symmetrization and get the desired result.

3.3. The general case. Here we prove Theorem 3.

3.3.1. *The graph.* Consider the graphs $G = G(n, m, \{a_0, a_1, \dots, a_m\}, x) = (V, E)$. We recall that the vertex set is

$$V = \Big\{ x = (x_1, \dots, x_n) \colon x_i \in \{0, 1, \dots, m\},\ |\{i \colon x_i = j\}| = a_j n \\ \forall\, j = 0, \dots, m,\ a_i \in (0, 1),\ \sum_{i=0}^{m} a_i = 1 \Big\}.$$

Two vectors are connected by an edge if and only if their scalar product is equal to xn.

As usual, we find the radius of the sphere containing a homothetical copy of our graph with edges of unit length. Let r' be the radius of the sphere containing our set. It is easy to see that the centre of this sphere lies in the hyperplane $\{\sum_{i=1}^{n} x_i = \sum_{i=0}^{m} i a_i n\}$ at the point $(\sum_{i=0}^{m} i a_i, \dots, \sum_{i=0}^{m} i a_i)$. Then

$$(r')^2 = \sum_{j=0}^{m} a_j n \Big(j - \sum_{i=0}^{m} i a_i \Big)^2 = \sum_{j=0}^{m} j^2 a_j n - 2 \sum_{j=0}^{m} j a_j \sum_{i=0}^{m} i a_i n + \sum_{j=0}^{m} a_j n \Big(\sum_{i=0}^{m} i a_i \Big)^2 \\ = \sum_{j=0}^{m} j^2 a_j n - 2 \Big(\sum_{j=0}^{m} j a_j \Big)^2 n + \Big(\sum_{i=0}^{m} i a_i \Big)^2 n = \sum_{j=0}^{m} j^2 a_j n - \Big(\sum_{j=0}^{m} j a_j \Big)^2 n.$$

The length l of any edge is given by the formula

$$l^2 = 2 \sum_{j=0}^{n} j^2 a_j n - 2xn.$$

Hence the squared radius of the sphere containing the normalized set is given by

$$r^2 = \frac{\sum_{j=0}^{m} j^2 a_j - \big(\sum_{j=0}^{m} j a_j \big)^2}{2 \big(\sum_{j=0}^{n} j^2 a_j - x \big)}.$$

3.3.2. *A geometric restriction on x.* We first obtain a restriction on x using a geometric argument (see Remark 9). Namely, the graph certainly contains no k-cliques if

$$\frac{k-1}{2k} > \frac{\sum_{j=0}^{m} j^2 a_j - \big(\sum_{j=0}^{m} j a_j \big)^2}{2 \big(\sum_{j=0}^{n} j^2 a_j - x \big)}.$$

This yields the following condition on x:

$$x < \frac{k \big(\sum_{j=0}^{m} j a_j \big)^2 - \sum_{j=0}^{m} j^2 a_j}{k-1}.$$

3.3.3. *Code restrictions on x.* Proceeding as usual, we form two equations. To get the first (resp. second) of these, we calculate the sum of all the coordinates in the clique (resp. the sum of the pairwise-scalar products in the clique) in two ways.

Let δ_i be the *degree of the* i*th coordinate*, that is, the sum of the ith coordinates of the vectors that form the k-clique. The first equation is obtained easily, but in the second it is not clear how to connect δ_i with the contribution d_i to the total scalar product. Suppose that the ith coordinate of our q_j^i vectors takes the value j, that is, $\sum_{j=0}^m q_j^i = k$ and $\sum_{j=0}^m jq_j^i = \delta_i$. What is the contribution of this coordinate to the total product? If we replace every vector with value j of this coordinate by j vectors with value 1 of the same coordinate, then the contribution of the resulting set exceeds the original contribution by $\frac{j(j-1)}{2}$. We get

$$d_i = \frac{\delta_i(\delta_i-1)}{2} - \sum_{j=0}^m \frac{j(j-1)}{2} q_j^i.$$

It is also known that $\sum_{i=1}^n q_j^i = ka_j n$. Thus we have the following system:

$$\begin{cases} \sum_{i=1}^n \delta_i = k\sum_{i=0}^m ia_in, \\ \sum_{i=1}^n 2d_i = k(k-1)xn \end{cases} \Leftrightarrow \begin{cases} \sum_{i=1}^n \delta_i = k\sum_{i=0}^m ia_in, \\ \sum_{i=1}^n \left(\delta_i(\delta_i-1) - \sum_{j=0}^m j(j-1)q_j^i\right) = k(k-1)xn \end{cases}$$

$$\Leftrightarrow \begin{cases} \sum_{i=1}^n \delta_i = k\sum_{i=0}^m ia_in, \\ \sum_{i=1}^n \delta_i(\delta_i-1) - k\sum_{j=0}^m j(j-1)a_jn = k(k-1)xn \end{cases}$$

$$\Leftrightarrow \begin{cases} \sum_{i=1}^n \delta_i = k\sum_{i=0}^m ia_in, \\ \sum_{i=1}^n \delta_i^2 - k\sum_{j=0}^m j^2a_jn = k(k-1)xn. \end{cases}$$

We minimize the sum of the squares of the numbers with a fixed sum in order to obtain the minimal x such that the corresponding graph can contain a k-clique for the fixed a_i in our approach.

The first and simplest way is to put $\delta_1,\dots,\delta_n = \delta = k\sum_{i=0}^m ia_i$. Then x takes the form

$$x = \frac{\left(k\sum_{j=0}^m ja_j\right)^2 - k\sum_{j=0}^m j^2a_j}{k(k-1)} = \frac{k\left(\sum_{j=0}^m ja_j\right)^2 - \sum_{j=0}^m j^2a_j}{k-1} = f_2.$$

Thus this approach yields exactly the same result as the geometric approach.

However, we can argue more carefully, in analogy with what was done in §§ 3.1,3.2. Namely, it is not always possible to make the degrees of all coordinates equal. The number $k\sum_{i=0}^m ia_i$ need not be an integer. Hence the degrees of the coordinates will take two values in the best case: there will be $\left\{k\sum_{i=0}^m ia_i\right\}n$ coordinates of degree $\delta+1 = \lceil k\sum_{i=0}^m ia_i\rceil$ and the remaining $\left(1-\left\{k\sum_{i=0}^m ia_i\right\}\right)n$ will have degree $\delta = \left[k\sum_{i=0}^m ia_i\right]$. Then the system takes the form

$$\begin{cases} \delta n + \left\{k\sum_{i=0}^m ia_i\right\}n = k\sum_{i=0}^m ia_i n, \\ \left(1-\left\{k\sum_{i=0}^m ia_i\right\}\right)n\delta^2 + \left\{k\sum_{i=0}^m ia_i\right\}n(\delta+1)^2 - k\sum_{j=0}^m j^2 a_j n = k(k-1)xn \end{cases}$$

$$\Leftrightarrow \begin{cases} \delta = \left[k\sum_{i=0}^m ia_i\right], \\ \delta^2 + \left\{k\sum_{i=0}^m ia_i\right\}(2\delta+1) - k\sum_{j=0}^m j^2 a_j = k(k-1)x \end{cases}$$

$$\Leftrightarrow \begin{cases} \delta = \left[k\sum_{i=0}^m ia_i\right], \\ \left(\left[k\sum_{i=0}^m ia_i\right] + \left\{k\sum_{i=0}^m ia_i\right\}\right)^2 - \left\{k\sum_{i=0}^m ia_i\right\}^2 + \left\{k\sum_{i=0}^m ia_i\right\} \\ \qquad - k\sum_{j=0}^m j^2 a_j = k(k-1)x. \end{cases}$$

The last system yields that

$$x = \frac{\left(k\sum_{i=0}^m ia_i\right)^2 - \left\{k\sum_{i=0}^m ia_i\right\}^2 + \left\{k\sum_{i=0}^m ia_i\right\} - k\sum_{j=0}^m j^2 a_j}{k(k-1)} = f_1.$$

Thus the graph contains no cliques if $x < f_1$. Comparing f_1 and f_2, we get

$$f_1 - f_2 = \frac{\left\{k\sum_{i=0}^m ia_i\right\} - \left\{k\sum_{i=0}^m ia_i\right\}^2}{k(k-1)}.$$

This proves the first part of Theorem 3.

3.3.4. *The construction.* Here we construct a clique in the graph $G = G(n, m, \{a_0, a_1, \dots, a_m\}, f_1)$ under certain additional restrictions on the parameters a_i and (as usual) on the divisibility of n. The restrictions on a_i are essential. In principle, one must improve the upper bound in Theorem 3 (as in parts 1, 2 of Theorem 2), but this leads to considerable difficulties.

Suppose that $\delta \leqslant k\sum_{i=0}^m ia_i < \delta + 1$. We first consider all possible non-negative integer solutions of the following system of equations:

$$\sum_{i=0}^m iq_i \in \{\delta, \delta+1\}, \qquad \sum_{i=0}^m q_i = k.$$

We enumerate them using a superscript j and denote them by $\mathbf{q}^j = (q_0^j, \dots, q_m^j)$.

Given a set of parameters $\mathbf{a} = (a_0, \dots, a_m)$, we assume that the vector $k\mathbf{a}$ lies in the convex hull of the vectors $\mathbf{q}^j$.

Thus,

$$k\mathbf{a} = \sum_j u_j \mathbf{q}^j, \tag{17}$$

where $\sum_j u_j = 1,\ u_j \geqslant 0$. Note that the components of $\mathbf{a}$ (resp. of the vectors $\mathbf{q}^j$) are rational (resp. integer) by assumption. Therefore one can also assume that all the u_j are rational. We denote the common denominator of the u_j by u and consider the sequence of dimensions $n_1' = un_1$, $n_2' = un_2$, … .

For brevity we write n, n' instead of n_i, n_i'. As in § 3.2.3, we start with an auxiliary construction. Namely, we write the linear combination (17) in terms of columns (the order of the entries in each column is arbitrary). This yields $u_j n'$ columns of the form $\mathbf{q}^j$ for every j (note that $u_j n'$ is an integer by definition).

We get a set of vectors where the degrees of the coordinates differ from each other by at most 1 and the total proportion of coordinates of type i is equal to a_i. It remains to symmetrize. In the sequence of dimensions $k!\,n_1'$, $k!\,n_2'$, … we construct a system of vectors by replacing each column of the auxiliary construction described above by the block of all the $k!$ permutations of the coordinates in that column. This preserves the properties already mentioned and also guarantees that each vector has the same number of the coordinates of every type and all the scalar products are pairwise equal. This proves that the estimate (4) is exact for some sequence of dimensions and some restrictions on the parameters.

To explain the assumption on $\mathbf{a}$ (about the convex hull), we give an example of explicit values of the parameters where this assumption on $\mathbf{a}$ can be omitted. Suppose that $a_i \geqslant \frac{1}{k}$, $i = 1, \dots, m$ (the condition holds for non-zero coordinates). We give only the auxiliary construction (the symmetrization will be the same). Since this construction is explicit, the divisibility conditions on n can be considerably weakened in this case.

We first sort all coordinates of all vectors in the clique according to value. The construction is in terms of rows.

Step 1. We put in rows all coordinates with value m. This process terminates at some point, and we get an uncompleted vector. We complete it by coordinates with value $m-1$ (this is possible since the total number of coordinates with value $m-1$ in k vectors is not less than n by assumption), and the step is done.

As a result, we are left with coordinate values $0, \dots, m-1$, and the degrees of the coordinate positions differ by at most 1. Further steps are made by induction.

We claim that at every step one can get rid of the corresponding largest coordinate value while leaving the differences between the degrees of the coordinate positions no greater than 1. Indeed, suppose that the ith step is done. Then the $(i+1)$th step is done as follows.

Step $i+1$. The largest value remaining is $m-i$. We consider all coordinates with this value and form as many vectors as possible from them. The remaining coordinates are then placed in those coordinate positions whose degree is the smaller of the two possible degrees. If some coordinates still remain, we place them in arbitrary remaining positions. Then we fill the remaining coordinates of the vector with coordinates with value $m-i-1$. It is easy to verify that our claim holds. Thus this process results in the desired auxiliary construction.

We note that the dimension is not required to be divisible by u here.

This completes the proof of Theorem 3.

§ 4. Proofs. Chromatic numbers

The proofs of Theorems 4, 5 are standard (see, for example, [19]). We outline them briefly for the sake of completeness.

4.1. Proof of Theorem 4. Consider a sequence of graphs $\mathcal{G} = \{G_n = G(n, 1, \{1-a_n, a_n\}, x_n)\}$ with the following conditions: $a_n \sim a$, $x_n \sim x$ and $a_n n, x_n n \in \mathbb{N}$ are such that $p_n \sim (a_n - x_n)n$, where p_n is a prime.

We note that for every x there is indeed a function $x(n)$ such that $(a-x(n))n = p$, where p is a prime, and $\frac{x}{x(n)} \sim 1$. This follows from the rather dense distribution of primes among the positive integers (there is always a prime between n and $n+\bar{o}(n)$; see [21]).

The following lemma plays a key role in the proof.

Lemma 2. *We have* $\alpha(G_n) = \alpha(G(n, 1, \{1-a_n, a_n\}, x_n)) \leqslant \sum_{j=0}^{p_n-1} \binom{n}{j}$. *Here* $\alpha(G)$ *is the size of a maximal independent set in the graph* G.

The standard bound for the chromatic number of any graph G states that $\chi(G) \geqslant \frac{|V|}{\alpha(G)}$.

We have $|V| = \binom{n}{a_n n}$, and Lemma 2 yields that $\alpha(G) \leqslant \sum_{j=0}^{p_n-1} \binom{n}{j} \leqslant n\binom{n}{p_n}$. We now use Stirling's formula:

$$\chi(G) \geqslant \frac{\binom{n}{a_n n}}{n\binom{n}{p_n}} = \left(\frac{\left(\frac{p_n}{n}\right)^{p_n/n}\left(1-\frac{p_n}{n}\right)^{1-p_n/n}}{a_n^{a_n}(1-a_n)^{1-a_n}} + \bar{o}(1)\right)^n$$
$$= \left(\frac{(a-x)^{a-x}(1-a+x)^{1-a+x}}{a^a(1-a)^{1-a}} + \bar{o}(1)\right)^n.$$

To complete the proof of Theorem 4, it remains to note that by the choice of the parameters a, x, Theorem 1 guarantees that the graph contains no cliques of size k for sufficiently large n. □

Proof of Lemma 2. For brevity we omit the subscripts n throughout. Consider an arbitrary set $Q = \{\mathbf{y}_1, \dots, \mathbf{y}_s\} \subset V$ with $(\mathbf{y}_i, \mathbf{y}_j) \neq xn$ for any distinct i, j. We want to prove that $s \leqslant \sum_{i=0}^{p-1} \binom{n}{i}$.

For every vector $\mathbf{y} \in V$ we define a polynomial $P_{\mathbf{y}} \in \mathbb{Z}/p\mathbb{Z}[z_1, \dots, z_n]$ by putting

$$P_{\mathbf{y}}(\mathbf{z}) = \prod_{i=0,\ i \not\equiv xn \pmod p}^{p-1} (i - (\mathbf{y}, \mathbf{z})), \qquad \mathbf{z} = (z_1, \dots, z_n).$$

Here the product is taken over all least non-negative residues i modulo p except for $i \equiv xn \pmod p$. Thus the degree of each $P_{\mathbf{y}}$ does not exceed $p-1$. Here is the most important property of these polynomials.

$(*)$ *For any* $\mathbf{y}, \mathbf{z} \in V$, *the equation* $(\mathbf{y}, \mathbf{z}) \equiv xn \pmod p$ *is equivalent to the inequality* $P_{\mathbf{y}}(\mathbf{z}) \not\equiv 0 \pmod p$.

Property $(*)$ is obvious and will be used without proof. For every $P_{\mathbf{y}}$ we construct another polynomial $P'_{\mathbf{y}}$ by writing $P_{\mathbf{y}}$ as a linear combination of the monomials

$$z_{i_1}^{\alpha_1} \cdots z_{i_q}^{\alpha_q}, \qquad q \leqslant p-1,$$

and replacing the α_ν by 1. The resulting monomials are just products of certain distinct variables. However, the new polynomials $P'_{\mathbf{y}}$, $\mathbf{y} \in V$, still possess property $(*)$ since we are considering variables that take the values 0, 1.

By the construction described, we have

$$\dim\bigl(\text{linear span}\{P'_{\mathbf{y}}\}_{\mathbf{y}\in V}\bigr) \leqslant \sum_{i=0}^{p-1} \binom{n}{i}.$$

The lemma will be proved once we can show that the vectors $\mathbf{y}_1, \dots, \mathbf{y}_s$ in the set Q correspond to a linear independent (over $\mathbb{Z}/p\mathbb{Z}$) set of polynomials $P'_{\mathbf{y}_1}, \dots, P'_{\mathbf{y}_s}$.

Assume that

$$c_1 P'_{\mathbf{y}_1}(\mathbf{z}) + \dots + c_s P'_{\mathbf{y}_s}(\mathbf{z}) \equiv 0 \pmod p \qquad \forall\, \mathbf{z} \in V. \tag{18}$$

Take $\mathbf{z} = \mathbf{y}_i$ for an arbitrary i. On the one hand, $(\mathbf{y}_i, \mathbf{y}_i) = an$. We have $an - p = xn$ by the hypotheses of Theorem 4. Therefore $(\mathbf{y}_i, \mathbf{y}_i) \equiv xn \pmod p$, whence $P'_{\mathbf{y}_i}(\mathbf{y}_i) \not\equiv 0 \pmod p$ by property $(*)$. On the other hand, if $j \ne i$, then $(\mathbf{y}_i, \mathbf{y}_j) < an$ and $(\mathbf{y}_i, \mathbf{y}_j) \ne xn$. Moreover, since $an - 2p < 0$ (because $a > 2x$), we have $(\mathbf{y}_i, \mathbf{y}_j) \not\equiv xn \pmod p$ and property $(*)$ yields that $P'_{\mathbf{y}_j}(\mathbf{y}_i) \equiv 0 \pmod p$.

By (18) we obtain $c_i \equiv 0 \pmod p$ (here it is important that p is a prime). Since i is arbitrary, it follows that the polynomials are linearly independent and the lemma is proved. □

4.2. Proof of Theorem 5. Consider a sequence of graphs $G_n = G'(n, \{b_n, 1 - a_n - b_n, a_n\}, x_n)$ with the following conditions: $a_n \sim a$, $b_n \sim b$, $x_n \sim x$ and $a_n n, b_n n, x_n n \in \mathbb{N}$ are such that $p_n \sim (a_n + b_n + x_n)n$, where p_n is a prime.

As in § 4.1, for every x there is a function $x(n)$ such that $(a + b + x(n))n = p$, where p is a prime, and $\frac{x}{x(n)} \sim 1$. Hence one can also find the desired sequences a_n, b_n, x_n.

The following lemma plays a key role in the proof of Theorem 5.

Lemma 3. *We have an inequality*

$$\alpha(G_n) = \alpha(G'(n, \{b_n, 1 - a_n - b_n, a_n\}, x_n)) \leqslant \sum_{j=0}^{[\frac{p_n-1}{2}]} \sum_{i=0}^{p_n-1-2j} \binom{n}{j}\binom{n-j}{i}.$$

Proof. This is similar to the proof of Lemma 2 (see [19]). □

Let us deduce Theorem 5 from Lemma 3. We have $\chi(G) \geqslant \frac{|V|}{\alpha(G)}$.

For brevity, we shall omit all the subscripts n and thus identify a_n with a, and so on. This is legitimate because $a_n \sim a$ and we are interested only in the limiting behaviour of the bounds with respect to n.

Since $|V| = \binom{n}{an}\binom{(1-a)n}{bn}$, we obtain by Lemma 3 that

$$\alpha(G) \leqslant \sum_{j=0}^{[\frac{p-1}{2}]} \sum_{i=0}^{p-1-2j} \binom{n}{j}\binom{n-j}{i} \leqslant n \sum_{j=0}^{[\frac{p-1}{2}]} \binom{n}{j}\binom{n-j}{p-1-2j} = nf.$$

We maximize the value of f with respect to j and denote it by $f_{\max}$. Clearly, $F < nf_{\max}$. To find the maximum of $f = \binom{n}{\alpha n}\binom{n-\alpha n}{p-2\alpha n}$, we recall that $p = p'n$ and use Stirling's formula:

$$f = \left(\frac{(1-\alpha)^{1-\alpha}}{\alpha^\alpha(1-\alpha)^{1-\alpha}(1+\alpha-p')^{1+\alpha-p'}(p'-2\alpha)^{p'-2\alpha}} + \bar{o}(1)\right)^n$$
$$= \left(\frac{1}{\alpha^\alpha(1+\alpha-p')^{1+\alpha-p'}(p'-2\alpha)^{p'-2\alpha}} + \bar{o}(1)\right)^n.$$

We now find the maximum (with respect to α) of the fraction in the brackets. Taking the logarithm and differentiating, we see that the maximum is attained at $\alpha = l = \frac{3p'+1-\sqrt{1+6p'-3(p')^2}}{6}$. It remains to take the expression corresponding to the number of vertices of the graph, expand it by the Stirling formula, and substitute in the formula $\chi(G) \geqslant \frac{|V|}{\alpha(G)}$. We get

$$\chi(G) \geqslant \frac{\binom{n}{an}\binom{(1-a)n}{bn}}{n^2 f_{\max}} \geqslant \left(\frac{l^l(p'-2l)^{p'-2l}(1-p'+l)^{1-p'+l}}{a^a b^b(1-a-b)^{1-a-b}} + \bar{o}(1)\right)^n.$$

Note that by our choice of parameters, Theorem 2 guarantees that the graph contains no cliques of size k for sufficiently large n. Theorem 5 is proved.

4.3. Proof of Theorem 6. We fix k, a (and therefore τ_0, τ_1, c). Theorem 6 holds trivially in the case when $c \leqslant 1$. Consider the situation when $c > 1$.

Take an arbitrary $c' \in (1, c)$. Here it is important that c' is strictly less than c, although it may be arbitrarily close to c. If we know that $\zeta_k \geqslant c'$, then, taking the supremum of both sides with respect to c', we obtain the desired inequality $\zeta_k \geqslant c$.

Thus we must establish the existence of a function $\delta(n) = \bar{o}(1)$ such that for every n there is a distance graph $G = (V, E)$ in $\mathbb{R}^n$ satisfying the inequalities $\omega(G) < k$ and $\chi(G) \geqslant \bigl(c' + \delta(n)\bigr)^n$.

We fix a, $0 < a < 1$. Let p be the smallest prime such that $[an] - 2p < 0$. For every sufficiently large $n \in \mathbb{N}$ we consider the graph $G_n = G(n, 1, \{1-a, a\}, p) = (V_n, E_n)$.

Using Stirling's formula, we easily see that

$$N = |V_n| = (\tau_1 + \bar{o}(1))^n.$$

At the same time, linear-algebraic methods yield the estimate (see Lemma 2)

$$\alpha = \alpha(G_n) \leqslant n\binom{n}{p}.$$

We have $p \sim \frac{a}{2}n$ as $n \to \infty$. Hence, by Stirling's formula, $\alpha \leqslant (\tau_0 + \bar{o}(1))^n$.

Of course, using the simple estimate $\chi(G) \geqslant \frac{|V|}{\alpha(G)}$, we find that

$$\chi(G_n) \geqslant \left(\frac{\tau_1}{\tau_0} + \delta_1(n)\right)^n$$

for some $\delta_1 = \bar{o}(1)$. The problem is that the distance graph G_n can contain cliques of size k.

We use the probabilistic method (see [22]) as in [18]. Consider an arbitrary number $\gamma \in (\frac{\tau_0 c'}{\tau_1}, 1)$. Such a γ exists since $c' < c \leqslant \frac{\tau_1}{\tau_0}$. Put $p = \gamma^n$, where n is sufficiently large. Construct a random subgraph $G = (V_n, E)$ in G_n by declaring that each edge of E_n occurs in E with probability p independently of the other edges (see [23]). We get a probability space $(\Omega_n, \mathcal{B}_n, P_n)$, where

$$\Omega_n = \{G = (V_n, E),\ E \subseteq E_n\}, \qquad \mathcal{B}_n = 2^{\Omega_n},$$
$$P_n(G) = q^{|E|}(1-q)^{|E_n|-|E|}, \qquad G = (V_n, E).$$

We define two families of events on Ω_n. First, for some l we enumerate all l-element subsets of V_n and introduce the events

$$X_i = \{\text{the } i\text{th } l\text{-element subset contains no edges}\}, \quad i = 1, \dots, \binom{N}{l}.$$

Second, we enumerate all k-cliques in G_n and introduce the events

$$Y_j = \{\text{the } j\text{th } k\text{-element subset is a clique in } G\}, \quad j = 1, \dots, \mathrm{cl}_k(G_n),$$

where $\mathrm{cl}_k(G_n)$ is the number of k-cliques in G_n.

Put $l = [(\frac{\tau_1}{c'})^n]$. Since $c' > 1$, we have $l < N = |\mathcal{V}_n|$ for large n and, therefore, the events X_i are well defined.

We must prove that

$$\mathsf{P}\biggl(\bigwedge_{i=1}^{\binom{N}{l}} \overline{X_i} \wedge \bigwedge_{j=1}^{\mathrm{cl}_k(G_n)} \overline{Y_j}\biggr) > 0. \tag{19}$$

Indeed, this will mean that there is a subgraph G' in G_n containing no k-cliques and satisfying $\alpha(G') \leqslant l$. Hence $\chi(G) \geqslant \bigl(c' + \bar{o}(1)\bigr)^n$, and Theorem 6 will be proved.

To establish (19), we recall Lovász' local lemma (see [22]).

Theorem 8 (Lovász' local lemma). *Let $A_1, \dots, A_m$ be events in some probability space, $J(1), \dots, J(m)$ subsets of $\{1, \dots, m\}$, and γ_i, $i = 1, \dots, m$, real numbers with $0 < \gamma_i < 1$ such that the following conditions hold.*

1) *A_i is independent of the algebra generated by the events $\{A_j,\ j \in \{1, \dots, m\} \setminus \{J(i) \cup \{i\}\}\}$.*
2) $\mathsf{P}(A_i) \leqslant \gamma_i \prod_{j \in J(i)} (1 - \gamma_j)$.

Then $\mathsf{P}\bigl(\bigwedge_{i=1}^m \bar{A}_i\bigr) \geqslant \prod_{i=1}^m (1 - \gamma_i) > 0$.

We use the following analogue of this result (see [23]).

Lemma 4. *Let $A_1, \dots, A_m$ and $J(1), \dots, J(m)$ be as in Theorem 8, and let δ_i be real numbers with $0 < \delta_i \mathsf{P}(A_i) < 0.69$, $i = 1, \dots, m$, such that*

$$\ln \delta_i \geqslant \sum_{j \in J(i)} 2\delta_j \mathsf{P}(A_j). \tag{20}$$

Then $\mathsf{P}\bigl(\bigwedge_{i=1}^m \bar{A}_i\bigr) \geqslant \prod_{i=1}^m (1 - \delta_i \mathsf{P}(A_i)) > 0$.

Proof. This can easily be deduced from Theorem 8. We only need to verify that the inequality in condition 2 of Theorem 8 follows from (20). Indeed, we have

$$\ln \delta_i \geqslant \sum_{j\in J(i)} 2\delta_j \mathsf{P}(A_j) \geqslant \sum_{j\in J(i)} -\ln(1-\delta_j \mathsf{P}(A_j))$$

since

$$\ln(1-t) \geqslant -t-t^2 \geqslant -2t \tag{21}$$

for $0<t<0.69$ (see [23]). Exponentiate both sides of (21):

$$\delta_i \geqslant \prod_{j\in J(i)} (1-\delta_j \mathsf{P}(A_j))^{-1}.$$

Substituting $\delta_i = \gamma_i/\mathsf{P}(A_i)$ proves the lemma. □

Before applying Lemma 4, we must calculate the probabilities of the events X_i, Y_i.

We start with X_i. Put $a_i = |E(G_n|_{W_i})|$, where W_i is the ith l-element subset of V_n. In other words, a_i is the number of edges inside W_i in the graph G_n. Then

$$\mathsf{P}(X_i) = (1-p)^{a_i} \leqslant e^{-pa_i} = e^{-\gamma^n a_i}.$$

Since $\frac{\tau_1}{c'} > \tau_0$, it is clear that $l > \alpha = \alpha(G_n)$ for all sufficiently large n. Therefore $a_i > 0$ for each W_i. One can easily establish that the much stronger inequality $a_i \geqslant \frac{l^2}{4\alpha}$ holds for $l > 2\alpha$.

Indeed, choose a maximal independent subset in the graph $G_n\big|_{W_i}$. It contains at most α elements. By maximality, each of the remaining $l-\alpha$ vertices is connected by an edge with a vertex in the independent set. We now choose a maximal independent subset of the remaining $l-\alpha$ vertices. There are at least $l-2\alpha$ edges from the vertices of this subset to the remaining $l-2\alpha$ vertices. We terminate this process at the first step s for which $l-s\alpha < \frac{l}{2}$. Then the total number of edges found during this process is at least $\frac{sl}{2}$, where $s > \frac{l}{2\alpha}$. Hence we have found at least $\frac{l^2}{4\alpha}$ edges.

Note that

$$\frac{l^2}{4\alpha} \geqslant \frac{\big((\frac{\tau_1}{c'})^2+\bar{o}(1)\big)^n}{(\tau_0+\bar{o}(1))^n} = \left(\frac{\tau_1^2}{\tau_0(c')^2}+\bar{o}(1)\right)^n. \tag{22}$$

We now calculate the probability of Y_i. It is easy to see that $\mathsf{P}(Y_i) = p^{k(k-1)/2} = \gamma^{nk(k-1)/2}$.

For each X_i we estimate the number of Y_j on which it depends. Note that these events are independent unless the corresponding sets W_i and Q_j have a common edge. There are at most $\operatorname{conn}_k^2(G_n)$ ways to construct a k-clique from a given edge (see (6)). Hence the number of Y_j on which X_i depends does not exceed $z_i = a_i N^{s_k^2(a,\frac{a}{2})+\bar{o}(1)}$.

Note that each Y_i depends on at most $\binom{N}{l}$ events of the form X_j. In a similar vein, each X_i depends on at most $\binom{N}{l}$ events of the form X_j.

It remains to estimate, for each Y_i, the number of events of the form Y_j on which Y_i depends. The dependence of two events means that they have a common edge. We easily see that the number of such dependences does not exceed $y = \binom{k}{2} N^{s_k^2(a,\frac{a}{2})+\bar{o}(1)} = N^{s_k^2(a,\frac{a}{2})+\bar{o}(1)}$.

For each X_i we divide the set $J(X_i)$ into two parts: the part $J^x(X_i)$ (resp. $J^y(X_i)$) contains all the events X_j (resp. Y_j) on which X_i depends. We similarly decompose the sets $J(Y_i)$. To apply Lemma 4, we rewrite the conditions (20) for the events X_i, Y_i respectively:

$$\begin{aligned} \ln \delta_i^x &\geqslant 2 \sum_{j\in J^x(X_i)} \delta_j^x e^{-\gamma^n a_j} + 2 \sum_{j\in J^y(X_i)} \delta_j^y \gamma^{nk(k-1)/2}, \\ \ln \delta_i^y &\geqslant 2 \sum_{j\in J^x(Y_i)} \delta_j^x e^{-\gamma^n a_j} + 2 \sum_{j\in J^y(Y_i)} \delta_j^y \gamma^{nk(k-1)/2}. \end{aligned} \tag{23}$$

Fix a function $\varepsilon = \varepsilon(\gamma)$ to be specified later, and put $\delta_i^y = e$, $\delta_i^x = e^{(\gamma-\varepsilon)^n a_i}$. We easily verify that $0 < \delta_i^x \mathsf{P}(X_i) < 0.69$ and $0 < \delta_i^y \mathsf{P}(Y_i) < 0.69$ for sufficiently large n (see Lemma 4). Then, for every j,

$$\begin{aligned} \delta_j^x e^{-\gamma^n a_j} &= e^{(\gamma-\varepsilon)^n a_j - \gamma^n a_j} = e^{-(\gamma-\bar{o}(1))^n a_j} \leqslant \exp\left\{-(\gamma-\bar{o}(1))^n \frac{l^2}{4\alpha}\right\} \\ &\overset{(22)}{\leqslant} \exp\left\{-\left(\frac{\gamma\tau_1^2}{(c')^2\tau_0} + \bar{o}(1)\right)^n\right\}. \end{aligned}$$

Furthermore,

$$\binom{N}{l} \leqslant \left(\frac{eN}{l}\right)^l \leqslant (c' + \bar{o}(1))^{\left(\frac{\tau_1}{c'}+\bar{o}(1)\right)^n} \leqslant \exp\left\{\left(\frac{\tau_1}{c'} + \bar{o}(1)\right)^n\right\}.$$

Thus, for every i, we have

$$\begin{aligned} \sum_{j\in J^x(X_i)} \delta_j^x e^{-\gamma^n a_j} &\leqslant \sum_{j\in J^x(X_i)} \exp\left\{-\left(\frac{\gamma\tau_1^2}{(c')^2\tau_0} + \bar{o}(1)\right)^n\right\} \\ &\leqslant \binom{N}{l} \exp\left\{-\left(\frac{\gamma\tau_1^2}{(c')^2\tau_0} + \bar{o}(1)\right)^n\right\} \\ &\leqslant \exp\left\{\left(\frac{\tau_1}{c'} + \bar{o}(1)\right)^n - \left(\frac{\gamma\tau_1^2}{(c')^2\tau_0} + \bar{o}(1)\right)^n\right\} = \bar{o}(1) \end{aligned}$$

since $\gamma > \frac{\tau_0 c'}{\tau_1}$ and, therefore, $\frac{\gamma\tau_1^2}{(c')^2\tau_0} > \frac{\tau_1}{c'}$. We similarly get

$$\sum_{j\in J^x(Y_i)} \delta_j^x e^{-\gamma^n a_j} = \bar{o}(1).$$

Thus, to verify that the inequalities (23) hold, it suffices to verify the following system of inequalities:

$$\begin{aligned} (\gamma-\varepsilon)^n a_i &\geqslant 2e a_i N^{s_k^2(a,\frac{a}{2})+\bar{o}(1)} \gamma^{nk(k-1)/2} \\ 1 &\geqslant 2e N^{s_k^2(a,\frac{a}{2})+\bar{o}(1)} \gamma^{nk(k-1)/2}. \end{aligned} \tag{24}$$

We easily see that it suffices to verify only the first inequality in (24). Hence, for every function $\delta(n)=\bar{o}(1)$ and all sufficiently large n we must have

$$1 \geqslant N^{s_k^2(a,\frac{a}{2})+\delta(n)}\gamma^{n\left(\frac{k(k-1)}{2}-1-\varepsilon_0\right)}, \tag{25}$$

where $\varepsilon_0=\log_\gamma(\gamma-\varepsilon)-1$ and, therefore, $\varepsilon=\varepsilon(\gamma)$ can be chosen in such a way that ε_0 is positive and arbitrarily small. Since

$$N^{s_k^2(a,\frac{a}{2})+\delta(n)}\gamma^{n\left(\frac{k(k-1)}{2}-1-\varepsilon_0\right)} = e^{n(\ln\tau_1)\left(s_k^2(a,\frac{a}{2})+\delta(n)\right)}e^{n(\ln\gamma)\left(\frac{k(k-1)}{2}-1-\varepsilon_0\right)},$$

it suffices to verify the following inequality instead of (25):

$$s_k^2\left(a,\frac{a}{2}\right)\ln\tau_1+\left(\frac{k(k-1)}{2}-1-\varepsilon_0\right)\ln\gamma<0. \tag{26}$$

Take γ to be very close to $\frac{\tau_0 c'}{\tau_1}$ so that

$$\gamma<\frac{\tau_0 c}{\tau_1}=\tau_1^{-\frac{2s_k^2(a,\frac{a}{2})}{(k-2)(k+1)}}$$

(this choice is possible because $c>c'$), and take ε_0 to be very small (in view of the choice of $\varepsilon(\gamma)$) so that

$$\gamma<\tau_1^{-\frac{2s_k^2(a,\frac{a}{2})}{(k-2)(k+1)-\varepsilon_0}}.$$

Substituting the last inequality in (26), we get

$$\begin{aligned}
&s_k^2\left(a,\frac{a}{2}\right)\ln\tau_1+\left(\frac{k(k-1)}{2}-1-\varepsilon_0\right)\ln\gamma\\
&\quad< s_k^2\left(a,\frac{a}{2}\right)\ln\tau_1+\left(\frac{(k+1)(k-2)}{2}-\varepsilon_0\right)\ln\left(\tau_1^{-\frac{2s_k^2(a,\frac{a}{2})}{(k-2)(k+1)-\varepsilon_0}}\right)=0.
\end{aligned}$$

Thus we have verified that all the hypotheses of Lemma 4 hold. Therefore inequality (19) holds and Theorem 6 is proved.

4.4. Proof of Theorem 7. This repeats the proof of Theorem 6 almost verbatim. The only difference is in the choice of the graph G_n.

Namely, we fix $a,b\in(0,1)$ with $a+b\leqslant\frac{1}{2}$, $a\geqslant b$. Let p be the smallest prime such that $[bn]+[an]-2p<-2[bn]$. Clearly, $p\sim\frac{3b+a}{2}$. Consider the graph $G_n=G'(n,\{b,1-a-b,a\},a+b-p)=(V_n,E_n)$ (see Definition 2).

By Stirling's formula, $|V_n|=(\rho_1+\bar{o}(1))^n$. It is also known that

$$\alpha=\alpha(\mathcal{G}_n)\leqslant(\rho_0+\bar{o}(1))^n$$

(see Lemma 3). Finally, we have an inequality of the form $r(W)\geqslant\frac{l^2}{4\alpha}$, $|W|=l\geqslant2\alpha$. The rest is clear and Theorem 7 is proved.

4.5. Proof of Proposition 2. The bound (7) and the inequality in (8) follow from Proposition 1.

To obtain the equality in (8), we must calculate the number of triangles with a given edge in the graph $G(n_i, 1, \{1-a, a\}, \frac{a}{2})$. Two vectors u, v in $G(n_i, 1, \{1-a, a\}, \frac{a}{2})$ that form an edge have $\frac{an}{2}$ common ones. For every t let $M(t)$ be the number of vectors that form a triangle with u, v and are such that the three vectors have tn common ones. By finding the maximum of $M(t)$ with respect to t, we shall see that the number of all vectors forming a triangle with a given edge does not exceed $\frac{an}{2}M = M(1+\bar{o}(1))^n$.

Indeed, it is easy to see that

$$M(t) = \binom{\frac{an}{2}}{tn}\binom{\frac{an}{2}}{\frac{an}{2}-tn}^2\binom{n-\frac{3an}{2}}{tn} = \binom{\frac{an}{2}}{tn}^3\binom{n-\frac{3an}{2}}{tn}$$
$$= \left(\frac{\left(1-\frac{3a}{2}\right)^{1-\frac{3a}{2}}\left(\frac{a}{2}\right)^{\frac{3a}{2}}}{t^{4t}\left(\frac{a}{2}-t\right)^{\frac{3a}{2}-3t}\left(1-\frac{3a}{2}-t\right)^{1-\frac{3a}{2}-t}} + \bar{o}(1)\right)^n.$$

It follows easily that

$$s_3^2\left(a, \frac{a}{2}\right) = \max_t \log_{\frac{1}{a^a(1-a)^{1-a}}}\left(\frac{\left(1-\frac{3a}{2}\right)^{1-\frac{3a}{2}}\left(\frac{a}{2}\right)^{\frac{3a}{2}}}{t^{4t}\left(\frac{a}{2}-t\right)^{\frac{3a}{2}-3t}\left(1-\frac{3a}{2}-t\right)^{1-\frac{3a}{2}-t}}\right).$$

Differentiating $M(t)$ with respect to t, we obtain that the maximum of $M(t)$ is attained at a root of the equation $t^4 = (\frac{1}{2}a - t)^3(1-\frac{3}{2}a - t)$. This equation can be solved, and its solution is as stated in the proposition.

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