On random subgraphs of a Kneser graph

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Kneser graph

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Kneser graph $KG_{n,k}$, $k \leqslant n/2$: vertices are k-element subsets of $[n]$, edges connect disjoint k-sets.

Introduced by M. Kneser in 1955, who conjectured that $\chi(KG_{n,k}) \leqslant n - 2k + 2$.

Conjecture was proved by L. Lovász in 1978 using topological methods.

Independence number (Erdős-Ko-Rado, 1961): $\alpha(KG_{n,k}) = \binom{n-1}{k-1}$.

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Chromatic number of Kneser-type graphs

Kneser graph $KG(\mathcal{A})$ for a system of k-sets $\mathcal{A} \subset \binom{[n]}{k}$: the vertices of $KG(\mathcal{A})$ are the elements of $\mathcal{A}$, edges connect disjoint k-sets.

Any such $KG(\mathcal{A})$ is an induced subgraph of $KG_{n,k}$.

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Random graph $\mathbf{KG_{n,k}(p)}$: the set of vertices is the same as for $KG_{n,k}$, each edge from $KG_{n,k}$ is included in $KG_{n,k}(p)$ with probability p.

L. Bogolyubskiy, A. Gusev, M. Pyaderkin and A. Raigorodskii studied $\alpha(KG_{n,k}(p))$. Raigorodskii, B. Bollobás: For some choice of parameters n, k, p it is equal to $\alpha(KG_{n,k}) \left(= \binom{n-1}{k-1}\right)$ w.h.p.

We study the chromatic number of Kneser graphs. For a wide range of parameters it is w.h.p. very close to $\chi(KG_{n,k})$:

Theorem (AK, 2014)

1. **If** p is fixed, $l \in \mathbb{N}$, and $k \gg n^{\frac{3}{2l}}$,
then w.h.p. $\chi(KG_{n,k}(p)) \geqslant \chi(KG_{n,k}) - 2l$.

2. **If** for some $p = p(n)$ we have $k \gg n^{3/4}p^{-1/4} + (n^{1/2}\ln n)p^{-1/2}$,
then w.h.p. $\chi(KG_{n,k}(p)) \geqslant \chi(KG_{n,k}) - 4$.

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Sketch of the proof. Part 1

Based on the proof of $\chi(KG_{n,k}) = n - 2k + 2$ by J. Greene.

Put $d = n - 2k - 2l + 1$. Roughly speaking, we show that in $KG_{n,k}$ there is a "small" amount of pairs of "big" subsets M^+, M^-, such that in any coloring of vertices of $KG_{n,k}$ in d colors one of the pairs form a monochromatic bipartite subgraph.

Fix a map from $[n]$ to the sphere S^d in general position (no $d+1$ points lie in a diametral sphere).

Estimate the probability of the following event A: for some diametral hyperplane π there are two "big" sets M^+, M^- in two opposite hemispheres such that there is no edge between M^+ and M^- in $KG_{n,k}(p)$.

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Show that A w.h.p. does not hold. Fix a graph for which the property A doesn't hold. Next, show that in any coloring of vertices of $KG_{n,k}$ in d colors there are two *monochromatic* sets M^+ and M^-.

Fix a coloring of vertices of $KG_{n,k}$ in d colors.

Construct an auxiliary covering of the sphere S^d by sets $B_0, \ldots, B_d$. Point x goes to the part B_i, $1 \leqslant i \leqslant d$, if in the open hemisphere with the center in x there are at least $k+l$ points of $[n]$ and color i is the most popular color in the coloring of the k-sets that lie wholly in that hemisphere.

If some colors i,j are equally popular, then add point x to both sets B_i, B_j.

All the rest goes to B_0.

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Theorem (Lusternik, Schnirelman 1930; Borsuk-Ulam 1933)

Whenever the sphere S^d is covered by sets $S_1, \dots, S_{d+1}$, each S_i is either open or closed, there exists i such that $S_i \cap (-S_i) \neq \emptyset$.

There are two antipodal points that are in the same set B_i. It cannot be B_0 because of the general position property.

Two sets of k-sets of color i in two opposite hemispheres form sets M^+, M^-. Since property A does not hold, there is an edge between them. Thus, the coloring is not proper.

The size of M^+, M^- we can get that way is at least $\binom{k+l}{k}/d$.

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- Can we prove w.h.p. $\chi(KG_{n,k}(p)) = \chi(KG_{n,k})$ for $p < 1$ and some $k = k(n)$?
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