On cliques in diameter graphs

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Diameter graphs

Diameter graph

A graph $G = (V, E)$ is a *diameter graph in* $\mathbb{R}^d$ *(or on* S_r^d*)* if $V \subset \mathbb{R}^d$ (S_r^d), V is finite, $\operatorname{diam} V = 1$ and

$$E \subseteq \{(x, y), x, y \in \mathbb{R}^d(S_r^d), |x - y| = 1\},$$

where $|x - y|$ denotes the Euclidean distance between x and y.

Borsuk's problem

Borsuk's problem: is it true that any bounded set in $\mathbb{R}^d$ can be partitioned into $d+1$ parts of strictly smaller diameter?

The finite version of Borsuk's problem

Is it true that any diameter graph G in $\mathbb{R}^d$ satisfies $\chi(G) \leq d+1$?

- 1955, H. Eggleston, true for $d = 3$.
- 1993, J. Kahn, G. Kalai, false for $d = 1325, d \geq 2016$.
- 2013, A. Bondarenko, false for $d \geq 65$.
- 2013, T. Jenrich, false for $d \geq 64$.

Main questions

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Theorem 1' (Kupavskii, AP 2013)

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How to obtain Theorem 1 out of Theorem 1'

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- Continue by induction.

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- Consider the $(d-1)$-dimensional sphere S with the center in the center of the clique K_1 and with radius of the circumscribed sphere of K_1.
- Case 1. All vertices of K_2 lie inside S. It's a simple case.
- Case 2. One vertex of K_2 lies outside S, all the rest lie inside S. What to do: rotate K_1 and reduce to the inductive assumption.

Remark on Maehara's results

Unit neighborhood graph

A graph $G = (V, E)$ is a **unit neighborhood graph** in $\mathbb{R}^d$ if $V \subset \mathbb{R}^d$ is a finite set and

$$E = \{xy,\ \text{where } x \neq y \in V, |x - y| \leqslant 1\}.$$

Sphericity

For a graph G, the **sphericity of G** (**sph G**) is the minimum dimension d such that G is *isomorphic* to a unit neighborhood graph in $\mathbb{R}^d$.

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Approach: consider circumscribed spheres and use induction.

Other conjectures and results

Conjecture 2 (Kupavskii, AP 2014)

Two cliques in a diameter graph in $\mathbb{R}^d$, one on $d+1$ vertices, the other on $\lfloor \frac{d+1}{2} \rfloor + 1$ vertices, either they share a vertex.

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Remark: $\lfloor \frac{d+1}{2} \rfloor + 1$ can't be replaced by $l = \lfloor \frac{d+1}{2} \rfloor$ in Conjecture 2.

Reuleaux simplex

A **Reuleaux simplex** Δ in $\mathbb{R}^d$ is a set formed by the intersection of the balls $B_i = B_1^d(v_i)$ of unit radius with centers in v_i, $i = 1, \ldots, d+1$, where v_i are the vertices of a unit simplex in $\mathbb{R}^d$.

Example to the remark: Consider midpoints of some l pairwise disjoint arcs that connect the vertices of the Reuleaux simplex.

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Theorem 3 (Kupavskii, AP 2014)

Conjecture 2 holds for $d = 4$.

Approach: divide the Reuleaux simplex into 3 parts and shift points that are inside of one of the parts.

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Problems

Problem 1 (Morić and Pach)

For a given d, characterize all pairs k, l of integers such that for any set of k red and l blue points in $\mathbb{R}^d$ we can choose a red point r and a blue point b such that $\|r - b\|$ is at least as large as (strictly greater than) the smallest distance between two points of the same color.

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Problem 1'

For a given d, characterize all pairs k, l of integers such that any k-clique and any l-clique in any diameter graph in $\mathbb{R}^d$ must share a vertex.