Unit distance graphs with no large cliques or short cycles and high chromatic number

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Two definitions

There are two well-known definitions of distance graphs. The first one is the following:

Complete distance graphs

We say that a graph $G = (V, E)$ is a *complete (unit) distance graph in* $\mathbb{R}^d$ if $V \subset \mathbb{R}^d$ and $E = \{(x, y), x, y \in \mathbb{R}^d, |x - y| = 1\}$.

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The second one is slightly different:

Distance graphs

We say that a graph $G = (V, E)$ is a *(unit) distance graph in* $\mathbb{R}^d$ if it is a subgraph of some complete distance graph in $\mathbb{R}^d$.

Motivation. Erdős on unit distances

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In 1965 Erdős, Harary and Tutte introduced the concept of the Euclidean dimension:

Euclidean dimension $\dim G$ of a graph G is the minimum dimension d so that the graph G is isomorphic to some distance graph in $\mathbb{R}^d$.

Motivation. Hadwiger-Nelson problem

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We can define analogous quantity in $\mathbb{R}^d$.
Formally,

$$\chi(\mathbb{R}^d) = \min\{m \in \mathbb{N} : \mathbb{R}^d = H_1 \cup \ldots \cup H_m : \\ \forall i, \forall x, y \in H_i \quad |x - y| \neq 1\}.$$

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Theorem(1951, P. Erdős, N.G. de Bruijn). If we accept the axiom of choice then the chromatic number of $\mathbb{R}^d$ is equal to the chromatic number of some *finite* distance graph in $\mathbb{R}^d$.

Large girth and large chromatic number

The *girth* of a graph the length of its shortest cycle.

Theorem (1959, P. Erdős). For any $l, k \in \mathbb{N}$ there exists a graph with chromatic number greater than l and with girth greater than k.

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Question: can we prove results of these type for distance graphs?

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In 2000 P. O'Donnell proved that

For any $k \in \mathbb{N}$ there exists a planar distance graph with the chromatic number equal to four and with girth larger than k.

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Theorem. We have

$$(\zeta_{low} + o(1))^n \leq \chi(\mathbb{R}^n) \leq (3 + o(1))^n, \text{ where } \zeta_{low} = 1.239\ldots$$

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Question. Whether there exists a sequence of distance graphs (complete distance graphs) in $\mathbb{R}^d$, $d = 1, 2, \ldots$, such that none of the graphs contain cliques of fixed size, and, additionally, the chromatic number of the graphs in the sequence grows exponentially with d?

What about graphs with girth greater than l for a fixed l greater than 3?

Formulation of the question

Consider the following four families of distance graphs in $\mathbb{R}^d$:

Denote by $\mathcal{C}(d,k)$ and $\mathcal{G}(d,k)$ the families of all distance graphs in $\mathbb{R}^d$ that do not contain k-cliques and have girth at least $k+1$ respectively. Similarly, define families $\mathcal{C}^*(d,k)$, $\mathcal{G}^*(d,k)$ of *complete* distance graphs.

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We define the following quantity:

$$\zeta_k = \liminf_{d\to\infty} \max_{G\in\mathcal{C}(d,k)} (\chi(G))^{1/d},$$

The quantities ζ_k^*, ξ_k and ξ_k^* are defined analogously, but here we maximize over the graphs from families $\mathcal{C}^*(d,k), \mathcal{G}(d,k)$ and $\mathcal{G}^*(d,k)$ respectively.

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Questions. Whether $\zeta_k > 1$ or not? What about $\zeta_k^*, \xi_k, \xi_k^*$? Is it true that $\zeta_k \geq c_k$, $\zeta_k^* \geq c_k$, where $c_k \to \zeta_{low}$ as $k \to \infty$?

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First, the number of distance and complete distance graphs differ greatly:

Theorem (AK, A. Raigorodskii, M. Titova; N. Alon, AK).
For any fixed d the number of distance graphs on n vertices in $\mathbb{R}^d$ is
$2^{\left(1-\frac{1}{[d/2]}+o(1)\right)\frac{n^2}{2}}$,
while the number of complete distance graphs is
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It is easy to see, that any bipartite graph is isomorphic to some graph from $\mathcal{G}(d,k)$, where $d \geq 4, k \geq 3$.
On the other hand, we have the following statement:

For any natural d there exists a bipartite graph that is not isomorphic to any complete distance graph in $\mathbb{R}^d$.

Quantities ζ_k, ζ_k^*

Raigorodskii and Rubanov showed that $\zeta_k > 1$ and that $\zeta_k \geq c_k$, where $c_k \to \zeta_{low}$ as $k \to \infty$.

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Two approaches to obtain bounds:

Probabilistic (Raigorodskii and Rubanov): no explicit graph, works only for ζ_k, we obtain $\zeta_k > 1$ only for $k \geq 5$.

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Two approaches to obtain bounds:

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Code-theoretic (Demechin, Raigorodskii and Rubanov): explicit constructions, works for $k \geq 3$ for both ζ_k and ζ_k^*. Better bounds for small k. But as k grows, the bounds tend to some constant that is smaller than ζ_{low}.

New results for ζ_k, ζ_k^*

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The refinement and generalization of code-theoretic approach allows us to improve all bounds on ζ_k and ζ_k^* based on this approach except for $k = 3$. We also prove that $\zeta_k^* \geq c_k$, where $c_k \to 1.154\ldots$ as $k \to \infty$.

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In total, we improve all bounds on ζ_k and ζ_k^* except for $k = 3$.

Question. Can we improve the constant $1.154\ldots$ for ζ_k^*?

Code-theoretic bounds

	old bounds	new bounds
k	$\zeta_k^* \geq$	$\zeta_k^* \geq$
3	1.0582	1.0582
4	1.0582	1.0663
5	1.0582	1.0857
6	1.0743	1.0898
7	1.0857	1.0995
8	1.0933	1.1019
9	1.0992	1.1077
10	1.1033	1.1093
11	1.1075	1.1131
12	1.1096	1.1145
13	1.1124	1.1175
$\lim_{k\to\infty}$	1.139	1.154

Probabilistic bounds

	old bounds	new bounds
k	$\zeta_k \geq$	$\zeta_k \geq$
3	—	1.0147
4	—	1.0321
5	1.0029	1.0491
6	1.0183	1.0641
7	1.0362	1.0771
8	1.0524	1.0881
9	1.0663	1.0976
10	1.0781	1.1057
11	1.0886	1.1128
12	1.0985	1.1190
13	1.1073	1.1245
14	1.1151	1.1293
15	1.1220	1.1336

Results for ξ_k, ξ_k^*

Theorem (AK). For any $k \in \mathbb{N}$ we have $\xi_k > 1$.

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Proposition (N. Alon, AK) For any $g \in \mathbb{N}$ there exists a sequence of complete distance graphs in $\mathbb{R}^d$, $d = 1, 2, \ldots,$ with girth greater than g such that the chromatic number of the graphs in the sequence grows as $\Omega\left(\frac{d}{\log d}\right)$.

The sketch of the proof. Part 1

The proof of the theorem is based on the analysis of the properties of the random subgraphs of the distance graphs $G_{4n} = (V_{4n}, E_{4n})$, where

$$V_{4n} = \{\mathbf{x} = (x_1, \ldots, x_{4n}) : \; x_i \in \{0, 1\}, \; x_1 + \ldots + x_{4n} = 2n\},$$

$$E_{4n} = \{\{\mathbf{x}, \mathbf{y}\} : \; (\mathbf{x}, \mathbf{y}) = n\}.$$

By $(,)$ we denote the scalar product.

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$$E_{4n} = \{\{\mathbf{x}, \mathbf{y}\} : \ (\mathbf{x}, \mathbf{y}) = n\}.$$

By $(,)$ we denote the scalar product.
Graphs of this type are used to obtain lower bounds on the chromatic number of the space.

The sketch of the proof. Part 2

It is easy to see that $|V_{4n}| = (2 + o(1))^{4n}, |E_{4n}| = (4 + o(1))^{4n}$.

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One of the main ingridients of the proof is the theorem by P. Frankl and V. Rödl concerning graphs G_{4n}:

Theorem

For any $\epsilon > 0$ there exists $\delta > 0$ such that for any subset S of V_{4n}, $|S| \geq (2 - \delta)^{4n}$, the number of edges in S (the cardinality of $E_{4n}|_S$) is greater than $(4 - \epsilon)^{4n}$.

The sketch of the proof. Part 3

Lovász local lemma

Let $A_1, \ldots, A_m$ be events in an arbitrary probability space and $J(1), \ldots, J(m)$ be subsets of $\{1, \ldots, m\}$. Suppose there are real numbers γ_i such that $0 < \gamma_i < 1,\ i = 1, \ldots, m$. Suppose the following conditions hold:

- A_i is independent of algebra generated by $\{A_j, j \notin J(i) \cup \{i\}\}$.
- $\mathrm{P}(A_i) \leq \gamma_i \prod_{j \in J(i)} (1 - \gamma_j)$.

Then $\mathrm{P}\left(\bigwedge_{i=1}^m \overline{A_i}\right) \geq \prod_{i=1}^m (1 - \gamma_i) > 0.$

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Then $\mathrm{P}\left(\bigwedge_{i=1}^m \overline{A_i}\right) \ge \prod_{i=1}^m (1 - \gamma_i) > 0.$

Using local lemma we prove that random subgraph of G_{4n} with positive probability does not contain cycles of length less than k and simultaneously the size of maximum independent set in the subgraph is not bigger than $(2-\epsilon)^{4n}$ for some $\epsilon > 0$.

Open problems

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2. Improve the bound on the chromatic number of sequences of complete distance graphs that have large girth.
3. Prove that for some r there exists a sequence of complete distance graphs that do not contain a copy of $K_{r,r}$ and whose chromatic number grows exponentially with the dimension.
4. Prove that for some k values of ζ_k, ζ_k^* (or ξ_k, ξ_k^*) are distinct.