*Sbornik*: *Mathematics* **204**:10 1435–1479 *Matematicheskiĭ Sbornik* **204**:10 47–90
DOI 10.1070/SM2013v204n10ABEH004345

# Obstructions to the realization of distance graphs with large chromatic numbers on spheres of small radii

A. B. Kupavskii and A. M. Raigorodskii

**Abstract.** We investigate in detail some properties of distance graphs constructed on the integer lattice. Such graphs find wide applications in problems of combinatorial geometry, in particular, such graphs were employed to answer Borsuk's question in the negative and to obtain exponential estimates for the chromatic number of the space.

This work is devoted to the study of the number of cliques and the chromatic number of such graphs under certain conditions. Constructions of sequences of distance graphs are given, in which the graphs have unit length edges and contain a large number of triangles that lie on a sphere of radius $1/\sqrt{3}$ (which is the minimum possible). At the same time, the chromatic numbers of the graphs depend exponentially on their dimension. The results of this work strengthen and generalize some of the results obtained in a series of papers devoted to related issues.

Bibliography: 29 titles.

**Keywords:** distance graph, chromatic number, clique, sphere of smallest radius.

## § 1. Introduction

By an *n-dimensional distance graph* (*n-dimensional graph of distances*) we mean any graph $G=(V,E)$ with

$$V\subseteq\mathbb{R}^n,\qquad E\subseteq\bigl\{\{\mathbf{x},\mathbf{y}\}:|\mathbf{x}-\mathbf{y}|=a\bigr\}$$

for some (arbitrary) $a>0$. Here $|\mathbf{x}-\mathbf{y}|$ is the conventional Euclidean distance between points. First, we shall distinguish between finite and infinite distance graphs, and second, we shall say that a distance graph is *complete* if in the definition of the set $E$ of its edges the inclusion '$\subseteq$' is replaced by exact equality. In other words, a graph is complete if it contains all possible edges having the given length $a$. Note that normalizing the set of vertices $V$ by dividing each of its elements by $a$ gives a graph which is isomorphic to the original one. In what follows we shall not distinguish between isomorphic graphs. Therefore, without loss of generality, we

This work was financially supported by the Russian Foundation for Basic Research (grant no. 12-01-00683), the Council for the Support of Leading Scientific Schools of the President of the Russian Federation (grant no. НШ-2519.2012.1), and the Council for the Support of Young Russian Doctors of Sciences of the President of the Russian Federation (grant no. МД-6277.2013.1).

*AMS* 2010 *Mathematics Subject Classification*. Primary 05D10, 05C15, 05C35.

shall usually assume that $a=1$. Such graphs are conventionally referred to as *unit distance graphs*.

There is a large body of literature devoted to distance graphs (see, for instance, [1]). One of the most important problems of combinatorial geometry which is intimately related to the concept of a unit distance graph is the problem of finding the so-called *chromatic number of the space* $\mathbb{R}^n$, which dates back to Nelson (see [2], [3]) and Hadwiger (see [4]), who investigated related issues somewhat earlier than Nelson. It is the quantity $\chi(\mathbb{R}^n)$, which is defined as the least number of colours needed to colour all points of $\mathbb{R}^n$ in such a way that no two of them distant from each other by the distance 1 are coloured the same colour:

$$\chi(\mathbb{R}^n)=\min\bigl\{\chi:\mathbb{R}^n=V_1\sqcup\dots\sqcup V_\chi,\ \forall\, i\ \forall\, \mathbf{x},\mathbf{y}\in V_i\ |\mathbf{x}-\mathbf{y}|\ne 1\bigr\}.$$

In terms of graphs this becomes the conventional *chromatic number* $\chi(\mathfrak{G})$ *of the graph* (see [5]) $\mathfrak{G}=(\mathbb{R}^n,\mathfrak{E})$, where

$$\mathfrak{E}=\{\{\mathbf{x},\mathbf{y}\}:|\mathbf{x}-\mathbf{y}|=1\}.$$

In other words, we consider the complete distance graph whose vertex set is the entire space $\mathbb{R}^n$.

Since in the Nelson-Hadwiger problem points coloured the same colour should not be distant from each other by the distance 1, in the context of this problem the value of 1 is referred to as the *forbidden distance*. Moreover, the same term is used for an arbitrary number $a$ involved in the definition of a distance graph. At the same time it is evident that the value of $a$ has no effect on the chromatic number of the space: as has already been mentioned, we shall not distinguish between isomorphic graphs, and from the viewpoint of colouring this is quite possible to do.

Chromatic numbers of spaces have been investigated deeply (see [1]–[12]). We shall be interested in estimates for them as $n\to\infty$. First of all we note that in what follows we shall often use relations of the form $f(n)\geqslant(2+o(1))^n$ for various functions $f(n)$. Each time it will mean that there exists a function $\delta(n)$ which tends to zero as $n\to\infty$ and, for all values of $n$, satisfies the estimate $f(n)\geqslant(2+\delta(n))^n$. Relations of the form $f(n)\leqslant n^2+O(n)$ and the like will be treated in the corresponding way. In this notation, chromatic numbers are known to satisfy the estimates

$$(\zeta_{\mathrm{low}}+o(1))^n\leqslant\chi(\mathbb{R}^n)\leqslant(\zeta_{\mathrm{high}}+o(1))^n,\qquad \zeta_{\mathrm{low}}=1.239\dots,\quad \zeta_{\mathrm{high}}=3.$$

The lower estimate was established by Raigorodskii [13], the upper bound by Larman and Rogers in [14]. It is significant that the lower bound is derived by considering a particular distance graph in $\mathbb{R}^n$. In this case it is a graph with vertices in $\{-1,0,1\}^n$. The previous (and the first ever obtained) exponential estimate for the chromatic number had the form

$$\chi(\mathbb{R}^n)\geqslant(\zeta'_{\mathrm{low}}+o(1))^n,\qquad \zeta'_{\mathrm{low}}=1.207\dots,$$

and was established by Frankl and Wilson using a graph with vertices in $\{0,1\}^n$ (see [15]). In turn, over the decade that has passed since [13] was published, graphs with vertices at arbitrary points of the lattice $\mathbb{Z}^n$ have been thoroughly investigated (see [16]–[18]).

Of particular interest are unit distance graphs of a given radius $r$ on spheres $S_r^{n-1} \subset \mathbb{R}^n$. Evidently, $r \geqslant 1/2$: otherwise, a graph with edges of length 1 cannot be drawn on a sphere. First, suppose that $r = 1/2$. Then it is clear that any graph $G$ on the corresponding sphere is bipartite (it is a matching, the edges of which join pairs of opposite points on the sphere). In other words, $\chi(G) \leqslant 2$. Erdős conjectured that $r = 1/2$ is the only value for which the chromatic number of any unit distance graph on the sphere of the corresponding radius is guaranteed to be finite (see [19]). The Erdős conjecture was proved by Lovaśz (see [20]) with the help of the topological method (see [21]): for any $r > 1/2$ and for any $n \in \mathbb{N}$ there exists a (finite) distance graph $G$ with edges of length 1 which lie on a sphere $S_r^{n-1}$ and whose chromatic number $\chi(G)$ is larger than $n-1$.

In recent works [22], [23] Raigorodskii employed the linear algebra method in combinatorics (see [8]) to significantly strengthen L. Lovaśz's results in the cases where $n \to \infty$. He succeeded in showing that for any $r > 1/2$ there exist a constant $\zeta_r > 1$, a function $\delta = \delta(n)$, which tends to zero with increasing $n$ (the dimension of the space), and a sequence of (finite) unit distance graphs $\{G_n\}_{n=1}^{\infty}$ such that $G_n \subset S_r^{n-1}$ and $\chi(G_n) \geqslant (\zeta_r + \delta)^n$.

On the one hand, the result mentioned above speaks well for the high stability of the property that the chromatic number of the space is exponential: it is preserved even when the entire space is limited to spheres of arbitrarily small (admissible) radii. Of course, $\zeta_r \to 1$ as $r \to 1/2$, and we still always have $\zeta_r > 1$.

On the other hand, it is known from the school curriculum that the circle circumscribed about the equilateral triangle with sides of length 1 has radius $1/\sqrt{3} = 0.577\ldots > 1/2$. In other words, the aforementioned result also means that there are sequences of distance graphs without triangles whose chromatic numbers grow exponentially. Moreover, there are sequences of graphs containing no cycles of odd length whose chromatic numbers are exponentially large (see [24]). (Note that cycles of even length exist on spheres of any radius $r > 1/2$.) In [24]–[27],

$$\zeta_{\text{clique}}(k) = \sup\bigl\{\zeta : \exists\,\delta(n),\ \delta(n) = o(1),\ \exists\,\{G_n\}_{n=1}^{\infty}, \\ G_n \subset \mathbb{R}^n,\ \omega(G_n) < k,\ \chi(G_n) \geqslant (\zeta + \delta(n))^n\bigr\},$$

$$\zeta_{\text{odd girth}}(k) = \sup\bigl\{\zeta : \exists\,\delta(n),\ \delta(n) = o(1),\ \exists\,\{G_n\}_{n=1}^{\infty}, \\ G_n \subset \mathbb{R}^n,\ g_{\text{odd}}(G_n) > k,\ \chi(G_n) \geqslant (\zeta + \delta(n))^n\bigr\},$$

the optimal constants in the bases of the corresponding exponentials, were investigated.

Here

$$\omega(G) = \max\bigl\{|W| : W \subseteq V,\ \forall\, x, y \in W\ \{x, y\} \in E\bigr\}$$

is the size of the largest *clique* (complete subgraph) in the graph $G = (V, E)$ and $g_{\text{odd}}(G)$ is the *odd girth* of the graph, that is, the length of the shortest odd cycle in the graph. For instance, each of the conditions $\omega(G) < 3$ and $g_{\text{odd}}(G) > 3$ means exactly that the graph $G$ contains no triangles.

Note that all the distance graphs that have been mentioned above have vertices in the normalized sets $\{0,1\}^n$, $\{-1,0,1\}^n$ and, more generally, in $\{0,1,\ldots,m\}^n$. In particular, $\lim_{k\to\infty} \zeta_{\text{clique}}(k) = \zeta_{\text{low}} = 1.239\ldots$, which was to be expected.

It follows from what was said above that the absence of cliques and short cycles in a graph serves as a weak constraint for the graph to have a large chromatic number. In this work we impose even stronger constraints on the class of graphs under consideration and show that even these constraints are, in a sense, 'almost uncorrelated' with the value of the chromatic number. We shall see that even on spheres of the smallest radius among all spheres that contain a clique of a given size there are distance graphs with a very large number of cliques and/or very large chromatic number. Moreover, the estimates for the chromatic numbers are almost the same as for the graphs which contain no cliques of size larger by one. This is evidence that cliques are in fact the only obstruction for the realization of distance graphs on spheres of a certain radius.

Exact statements of the problems are given in the next section. Since cliques are not that easy to deal with, and cycles are even more difficult to handle, and dealing with cycles requires additional ideas and methods, we shall investigate cycles in a separate work.

## § 2. Statements of the problems

It has already been mentioned that the circle circumscribed about an equilateral triangle with sides of length 1 has radius $r = 1/\sqrt{3}$. A more general fact is that the radius of the sphere circumscribed around a regular simplex with $k$ vertices (that is, a simplex of dimension $k-1$) equals $\sqrt{(k-1)/(2k)}$. For $k=3$ we have exactly $1/\sqrt{3}$. Let us set $r_{\text{clique}}(k) = \sqrt{(k-1)/(2k)}$. This is the smallest radius of a sphere on which one can draw a unit distance graph containing a clique of size $k$.

Of course, there exist distance graphs $G$ satisfying the conditions $\omega(G) = k$ and $G \subset S^{n-1}_{r_{\text{clique}}(k)}$ for $n \geqslant k$. For instance, one can take just one simplex with $k$ vertices. The problem is to find out if there exist distance graphs with a large number of cliques and/or a large chromatic number of spheres of such a small — critically small — radius.

Let $N(G)$ be the number of vertices in a graph $G$. Denote by $X_k(G)$ the number of $k$-cliques in $G$. Clearly, we have

$$X_k(G) \leqslant C^k_{N(G)} = O(N^k(G))$$

(we assume everywhere that $k$ is a constant). It is also easy to present a graph $G \subset S^{n-1}_{r_{\text{clique}}(k)}$ with $X_k(G) = \Omega(N(G))$. It suffices to arrange $n$ mutually disjoint simplices of dimension $k-1$ on a sphere. However, first, $N(G)$ is incomparably less than $N^k(G)$, and second, there is not much to be said about the chromatic number of such graphs: it is equal to $k$, which means that it does not even grow as $n \to \infty$. Leaving the problem of the chromatic number aside for a while, we consider the quantity

$$\kappa_{\text{clique}}(k) = \max\bigl\{\kappa : \exists\, \varepsilon(n),\ \varepsilon(n) = o(1),\ \exists\, \{G_n\}_{n=1}^{\infty}, \\ G_n \subset S^{n-1}_{r_{\text{clique}}(k)},\ N(G_n) \nearrow \infty,\ X_k(G_n) = \Omega(N^{\kappa+\varepsilon(n)}(G_n))\bigr\}.$$

In view of the remarks made above it is obvious that $1 \leqslant \kappa_{\text{clique}}(k) \leqslant k$, and it is by no means evident that $\kappa_{\text{clique}}(k) > 1$. Nevertheless, in the subsequent sections we shall demonstrate that $\kappa_{\text{clique}}(k)$ almost attains its maximum possible value with increasing $k$. Note that the function $\varepsilon$ involved in the definition of $\kappa_{\text{clique}}(k)$ is introduced for the reason that we shall often obtain estimates of the form

$$X_k(G_n) = \Omega\biggl(\frac{N^{\beta}(G_n)}{\ln^{\gamma}(N(G_n))}\biggr), \qquad \gamma > 0, \quad \beta > 1,$$

in which we cannot get rid of logarithms. It is in order not to deal with logarithms and similar 'add-ons' that we introduce the correction $\varepsilon$.

Now we return to the problem of the chromatic number. Suppose that we have constructed a distance graph $G$ on the sphere $S^{n-1}_{r_{\text{clique}}(k)}$ with some number of cliques (for instance, the maximum possible number). In view of the results in [22], [23], on the same sphere one can arrange a graph $H$ with the chromatic number

$$\chi(H) \geqslant (\zeta_{r_{\text{clique}}(k)} + \delta)^n$$

(see § 1). If the value of $N(G)$ is not much different from $N(H)$, then taking the union of the graphs $G$ and $H$ we obtain a graph $F$ with a large number of cliques and a large chromatic number at the same time.

Even if we forget that $N(G) \approx N(H)$, the example given above seems quite artificial. Therefore, we shall not take a close look at the condition of the approximate equality between the numbers of vertices, but recall what has already been discussed several times in the introduction: all lower bounds for various interesting 'colour' characteristics are derived with the use of $\{0,1\}$-, $\{-1,0,1\}$- and $\{0,1,\dots,m\}$-points. Moreover, the inequalities announced above for $\kappa_{\text{clique}}(k)$ will also be substantiated with the use of such points. Therefore, it is reasonable to consider specializations of our problem to graphs with vertices located at points of the lattice $\mathbb{Z}^n$. Here we are faced with a small difficulty. Of course, such graphs do not lie on the spheres $S^{n-1}_{r_{\text{clique}}(k)}$ and (as a rule) are not unit distance graphs. However, after an appropriate normalization their homothetic copies may well both be arranged on the spheres and have edges of length 1. In what follows, when talking about the 'cases of $\{0,1,\dots,m\}$-points' we shall assume that they are appropriately normalized. Note that the case of $\{-1,0,1\}$-points can be reduced to the case of $\{0,1,2\}$-points by a simple parallel translation. Therefore, it also agrees with the general approach for $m=2$.

Thus, we say that an $n$-dimensional distance graph $G=(V,E)$ *belongs to the class* $\mathscr{A}(n,m)$ if $V \subset \{0,1,\dots,m\}^n$ and for any $\mathbf{x} \in V$, $\mathbf{x}=(x_1,\dots,x_n)$, and any $i \in \{0,1,\dots,m\}$, the number of coordinates $x_j$ equal to $i$ is the same. In other words, there are positive integers $l_0,l_1,\dots,l_m$ which sum up to $n$ and satisfy the conditions

$$|\{j : x_j = 0\}| = l_0, \qquad |\{j : x_j = 1\}| = l_1, \qquad \dots, \qquad |\{j : x_j = m\}| = l_m.$$

It is obvious that a graph $G$ with particular $l_0,l_1,\dots,l_m$ automatically lies on a sphere, the squared radius of which equals $l_1 + 4l_2 + \dots + m^2 l_m$.

We say that an $n$-dimensional distance graph $G=(V,E)$ *belongs to the class* $\mathscr{A}(n,m,r)$, $r \geqslant 1/2$, if $G$ is a unit distance graph, $G \subset S^{n-1}_r$, and $G$ is obtained from some graph $H \in \mathscr{A}(n,m)$ via normalization. Since for $m=2$ it will be convenient to consider $\{-1,0,1\}$-points rather than $\{0,1,2\}$-points, in order to avoid confusion we introduce two more classes: $\mathscr{A}'(n,2)$ and $\mathscr{A}'(n,2,r)$.

Now let us define several new 'extremal' quantities. First of all, we set

$$\begin{aligned}\kappa_{\text{clique}}(k,m)=\max\bigl\{&\kappa:\exists\,\varepsilon(n),\ \varepsilon(n)=o(1),\ \exists\,\{G_n\}_{n=1}^{\infty},\\ &G_n\in\mathscr{A}(n,m,r_{\text{clique}}(k)),\ N(G_n)\nearrow\infty,\ X_k(G_n)=\Omega(N^{\kappa+\varepsilon(n)}(G_n))\bigr\}.\end{aligned}$$

It is clear that $\kappa_{\text{clique}}(k)\geqslant\max_m\kappa_{\text{clique}}(k,m)$. In this work, it is the last maximum that we shall be interested in.

Further, with graphs that are in the classes $\mathscr{A}(n,m,r)$ the trick of the artificial unification of a graph $G$ having a large value of $X_k$ and a graph $H$ having a large value of $\chi$ does not work. Of course, both graphs belong to some classes $\mathscr{A}(n,m,r)$, but $G$ has its own parameters $l_0,l_1,\dots,l_m$, which are in general different from the parameters of $H$. Therefore, it becomes reasonable to introduce and investigate the following characteristics:

$$\begin{aligned}\kappa^{\chi}_{\text{clique}}(k,m)=\max\bigl\{&\kappa:\exists\,\varepsilon(n),\ \varepsilon(n)=o(1),\ \exists\,\{G_n\}_{n=1}^{\infty},\ G_n\in\mathscr{A}(n,m,r_{\text{clique}}(k)),\\ &N(G_n)\nearrow\infty,\ X_k(G_n)=\Omega(N^{\kappa+\varepsilon(n)}(G_n)),\ \chi(G_n)\to\infty\bigr\},\\ \kappa^{\chi-\exp}_{\text{clique}}(k,m)=\max\bigl\{&\kappa:\exists\,\varepsilon(n),\ \varepsilon(n)=o(1),\ \exists\,\delta(n),\ \delta(n)=o(1),\\ &\exists\,\zeta>1,\ \exists\,\{G_n\}_{n=1}^{\infty},\ G_n\in\mathscr{A}(n,m,r_{\text{clique}}(k)),\ N(G_n)\nearrow\infty,\\ &X_k(G_n)=\Omega(N^{\kappa+\varepsilon(n)}(G_n)),\ \chi(G_n)\geqslant(\zeta+\delta(n))^n\bigr\},\\ \kappa^{\chi}_{\text{clique}}(k)=\max_m&\,\kappa^{\chi}_{\text{clique}}(k,m),\qquad \kappa^{\chi-\exp}_{\text{clique}}(k)=\max_m\kappa^{\chi-\exp}_{\text{clique}}(k,m),\\ \chi^{\kappa}_{\text{clique}}(k,m)=\sup\bigl\{&\chi:\exists\,\varepsilon(n),\ \varepsilon(n)=o(1),\ \ \exists\,\delta(n),\ \delta(n)=o(1),\\ &\exists\,\kappa>1,\ \exists\,\{G_n\}_{n=1}^{\infty},\ G_n\in\mathscr{A}(n,m,r_{\text{clique}}(k)),\ N(G_n)\nearrow\infty,\\ &X_k(G_n)=\Omega(N^{\kappa+\varepsilon(n)}(G_n)),\ \chi(G_n)\geqslant(\chi+\delta(n))^n\bigr\},\\ \chi^{\kappa}_{\text{clique}}(k)=&\max_m\chi^{\kappa}_{\text{clique}}(k,m).\end{aligned}$$

Note that the mere existence of graphs in the classes $\mathscr{A}(n,m,r)$ for arbitrary values of $r$ is not that obvious. In turn, we shall be interested in sequences of distance graphs in the classes $\mathscr{A}(n,m,r_{\text{clique}}(k))$, that is, sequences whose elements lie on spheres of *critical* radius. Moreover, these elements (graphs) should satisfy the following additional conditions:

- they contain as many cliques as possible (the quantities $\kappa_{\text{clique}}(k,m)$, $\kappa_{\text{clique}}(k)$);
- they contain as many cliques as possible under the condition that the chromatic number grows infinitely (the quantities $\kappa^{\chi}_{\text{clique}}(k,m)$, $\kappa^{\chi}_{\text{clique}}(k)$);
- they contain as many cliques as possible under the condition that the chromatic number grows exponentially (the quantities $\kappa^{\chi-\exp}_{\text{clique}}(k,m)$, $\kappa^{\chi-\exp}_{\text{clique}}(k)$);
- they have the largest possible chromatic number under the condition that there are untrivially many cliques in them (the quantities $\chi^{\kappa}_{\text{clique}}(k,m)$, $\chi^{\kappa}_{\text{clique}}(k)$).

In § 3 we shall discuss the case $m=1$, in § 4, the case $m=2$, and in § 5, the general case.

## § 3. Constructions with $\{0,1\}$-points

In this section we first learn how to handle the case $k=3$, that is, we study triangles on spheres. Then we pass to the case of arbitrary cliques. Each time, we employ graphs belonging to $\mathscr{A}(n,1)$. All graphs considered here are complete.

The section is divided into four subsections. In § 3.1 we describe the properties of graphs belonging to the class $\mathscr{A}(n,1)$ and, in particular, calculate the radii of the spheres on which these graphs are located. In § 3.2, which is in turn divided into subsubsections, we study triangles. In § 3.3, which is divided further as well, we investigate $k$-cliques with $k\geqslant 4$. In § 3.4 we briefly review the main results of this section.

**3.1. General properties of a graph in the class $\mathscr{A}(n,1)$.** If a graph $G$ belongs to the class $\mathscr{A}(n,1)$, then each of its vertices has $l_1$ coordinates equal to 1 and $l_0=n-l_1$ zero coordinates. Denote by $y_1$ the value of the forbidden distance that is responsible for the edges of $G$. Since all vertices of the graph have the same number of coordinates of a given value, the forbidden distance $y_1$ is uniquely expressed in terms of the forbidden inner product $x_1$: $|\mathbf{x}-\mathbf{y}|^2=2l_1-2(\mathbf{x},\mathbf{y})$, so that $y_1^2=2l_1-2x_1$. As a result, the properties of the graph $G$ are determined not only by the value of $n$, but also by the quantities $l_1\in\{0,\dots,n\}$ and $x_1\in\{0,\dots,l_1\}$.

We find the radius of the sphere on which $G$ lies. Of course, we know that $G\subset S_r^{n-1}$ with $r^2=l_1$. However, it is easily seen that $G$ lies in the intersection of the aforementioned sphere with the plane $\{\mathbf{x}=(x_1,\dots,x_n): x_1+\dots+x_n=l_1\}$, and hence, $G$ lies on a sphere centred at $(l_1/n,\dots,l_1/n)$. Thus, the desired radius $r=r(n,l_1)$ is calculated by the formula

$$r^2=l_1\biggl(1-\frac{l_1}{n}\biggr)^2+(n-l_1)\frac{l_1^2}{n^2}.$$

Let us see what happens if we reduce graph $G$ to a unit distance graph via an appropriate normalization. Of course, we must scale by $y_1$. As a result, the graph $H=(1/y_1)\cdot G$ occurs on the sphere $S_{r'}^{n-1}$ with

$$r'=r'(n,l_1,x_1)=\sqrt{\frac{l_1(1-l_1/n)^2+(n-l_1)l_1^2/n^2}{2l_1-2x_1}}. \tag{3.1}$$

In other words, $H\in\mathscr{A}(n,1,r')$.

Formula (3.1) does not seem to be suitable for calculations. Let us represent $l_1$ in the form $l_1=an$ and $x_1$ in the form $x_1=xn$. Of course, such a change of variables makes the formulae simpler: indeed, as a result of such a representation, after appropriate cancellations we have

$$r'=\sqrt{\frac{a(1-a)}{2a-2x}}. \tag{3.2}$$

At the same time, this change of variables is quite restrictive, since from now on we have to assume that $a,x\in\mathbb{Q}$, and hence $n$ is such that $an\in\mathbb{N}$ and $xn\in\mathbb{N}$ (this is what the restriction caused by the above change of variables consists of). Otherwise, the formula is simply not correct.

However, we see already that as soon as we estimate 'extremal' characteristics of the form $\kappa_{\text{clique}}(k,1)$ and $\kappa_{\text{clique}}(k)$, it is much more convenient to optimize over $a \in [0,1]$ and $x \in [0,a]$, rather then over arbitrary $l_1$ and $x_1$. In § 3.2 we show that in doing so we do not lose much: in a sense, we lose nothing at all.

**3.2. Triangles.** In this subsection we investigate the quantities $\kappa_{\text{clique}}(3,1)$, $\kappa_{\text{clique}}(3)$ and others. The subsection is divided into subsubsections. In § 3.2.1 we consider a rather special case of the construction described in § 3.1 and derive some estimates for the quantities $\kappa_{\text{clique}}(3,1)$ and $\kappa_{\text{clique}}(3)$. In § 3.2.2 we make some important intermediate comments on the results obtained. In § 3.2.3 we prove the optimality of the construction suggested in § 3.2.1. Finally, in § 3.2.4 we discuss the chromatic numbers of the graphs mentioned in the previous subsubsections.

3.2.1. *The case $a=1/3$, $x=0$.* In the case where $a=1/3$, $x=0$, we have

$$r' = r'(n,l_1,x_1) = r'(n,a,x) = \sqrt{\frac{1/3\cdot 2/3}{2/3}} = \frac{1}{\sqrt{3}} = r_{\text{clique}}(3).$$

In other words, if $n$ is divisible by 3, then the corresponding graph $G_n$ belongs to the class $\mathscr{A}(n,1,r_{\text{clique}}(3))$. For such a graph we have

$$N(G_n) = C_n^{an} = C_n^{n/3} = \left(\frac{1}{(1/3)^{1/3}(2/3)^{2/3}} + o(1)\right)^n = (1.889\ldots + o(1))^n \nearrow \infty.$$

Further, triangles in $G_n$ are formed by vertices such that all the pairwise inner products of the corresponding vectors vanish (which is to say that the subsets of their nonzero coordinates are pairwise disjoint[1]). Then it is clear that the number of triangles in $G_n$ has the form

$$X_3(G_n) = C_n^{n/3} C_{2n/3}^{n/3} = (3+o(1))^n.$$

As a result,

$$X_3(G_n) = \Omega(N^{\log_{(((1/3)^{1/3}(2/3)^{2/3})^{-1}+o(1))}(3+o(1))}(G_n)) = \Omega(N^{1.726\ldots+o(1)}(G_n)).$$

Looking at the definition of the quantity $\kappa_{\text{clique}}(3,1)$, one might think that we have just derived the estimate $\kappa_{\text{clique}}(3,1) \geqslant 1.726\ldots$. Unfortunately, this is not exactly so. Indeed, we have obtained an $\varepsilon = o(1)$ and a sequence of graphs $G_n \in \mathscr{A}(n,1,r_{\text{clique}}(3))$ satisfying the properties $N(G_n) \nearrow \infty$ and

$$X_3(G_n) = \Omega(N^{1.726\ldots+\varepsilon}(G_n)).$$

The problem is that here $n$ runs not through all positive integers, but only multiples of 3. Properly speaking, we should introduce the new quantity

$$\begin{aligned}\widehat{\kappa}_{\text{clique}}(k,m) = \max\bigl\{\kappa : \exists\,\varepsilon(n),\ \varepsilon(n)=o(1),\ \exists\,\{n_i\}_{i=1}^{\infty},\ \exists\,\{G_{n_i}\}_{i=1}^{\infty},\\ G_{n_i} \in \mathscr{A}(n_i,m,r_{\text{clique}}(k)),\ N(G_{n_i}) \nearrow \infty,\ X_k(G_{n_i}) = \Omega(N^{\kappa+\varepsilon(n_i)}(G_{n_i}))\bigr\}.\end{aligned}$$

It is this quantity and the corresponding quantity $\widehat{\kappa}_{\text{clique}}(k)$ that satisfy the following bound.

[1] We say 'nonzero', rather than 'unit', because, strictly speaking, the graph $G_n$ is obtained by normalizing some graph on $\{0,1\}$-points.

**Theorem 1.** *The following estimate holds*:

$$\widehat{\kappa}_{\text{clique}}(3) \geqslant \widehat{\kappa}_{\text{clique}}(3,1) \geqslant \log_{\frac{1}{(1/3)^{1/3}(2/3)^{2/3}}} 3 = 1.726\dots.$$

In the next subsubsection we shall discuss how strong the restriction that we had to make actually is.

3.2.2. *Intermediate comments.* In the previous subsubsection we proved the simple Theorem 1. Unfortunately, the bound established in that theorem applies not to the original quantities, but to the new 'hatted' quantities. In this subsubsection we touch upon the question of whether the problem encountered can or should be fixed.

One may attempt to overcome this difficulty, for instance, in the following way: replace the quantity $an$ in the definition of the graph in § 3.1 with the quantity $[an]$. Of course, all calculations made in § 3.2.1 remain valid (for $a = 1/3$). However, as a matter of fact, we reduce the convenient formula (3.2) back to the cumbersome formula (3.1). Moreover, instead of the exact equality $r' = 1/\sqrt{3}$ we establish only the asymptotic behaviour $r' \sim 1/\sqrt{3}$. This is definitely not a resolution of the problem.

In fact, the situation is much better.

**Theorem 1′.** *The following estimate holds*:

$$\kappa_{\text{clique}}(3) \geqslant \kappa_{\text{clique}}(3,1) \geqslant \log_{\frac{1}{(1/3)^{1/3}(2/3)^{2/3}}} 3 = 1.726\dots.$$

*Proof.* Let $n$ be an arbitrary positive integer. Take the greatest positive integer $\nu$ which is divisible by 3 and does not exceed $n$. It is evident that $\nu \geqslant n-2$. We know from § 3.2.1 that there exists a graph $G_\nu \in \mathscr{A}(\nu, 1, r_{\text{clique}}(3))$ with

$$N(G_\nu) = \biggl(\frac{1}{(1/3)^{1/3}(2/3)^{2/3}} + \delta_1(\nu)\biggr)^{\nu}, \qquad X_3(G_\nu) = (3+\delta_2(\nu))^{\nu},$$
$$\delta_1 = o(1), \quad \delta_2 = o(1), \qquad \nu \to \infty.$$

We add to the end of the coordinates of each vertex of the graph $G_\nu$ $n-\nu$ zero coordinates (the edges being preserved) and denote the graph obtained by $G_n$. It is clear that $G_n \in \mathscr{A}(n, 1, r_{\text{clique}}(3))$ and that

$$N(G_n) = \biggl(\frac{1}{(1/3)^{1/3}(2/3)^{2/3}} + \delta_1(\nu)\biggr)^{\nu} = \biggl(\frac{1}{(1/3)^{1/3}(2/3)^{2/3}} + \delta_1'(n)\biggr)^{n},$$
$$\delta_1' = o(1), \qquad n \to \infty,$$
$$X_3(G_n) = (3+\delta_2'(n))^{n}, \qquad \delta_2' = o(1), \quad n \to \infty.$$

Thus, we have once again

$$X_3(G_n) = \Omega\bigl(N^{\log_{(((1/3)^{1/3}(2/3)^{2/3})^{-1}+o(1))}(3+o(1))}(G_n)\bigr) = \Omega\bigl(N^{1.726\ldots+o(1)}(G_n)\bigr),$$

which proves the theorem.

Throughout this work (for instance, in the case of $\{-1,0,1\}$-points), the corresponding theorems 'with primes' can be established. In order not to do it each time, we replace all quantities in the second subsection with the corresponding 'hatted' quantities (as before the formulation of Theorem 1) and study only these 'hatted' quantities. However, in doing so we keep in mind that the results obtained for the 'hatted' quantities are usually easy to adjust to the original quantities (without hats).

3.2.3. *The estimate established in Theorem 1 is optimal.* In this subsubsection we find all pairs $a$, $x$ such that any graph in the class $\mathscr{A}(n,1,r')$ ($r'$ is determined by formula (3.2)) contains triangles and the value of $r'$ is the smallest (that is, $r'=r_{\text{clique}}(3)$). Eventually we shall see that the pair $(a,x)=(1/3,0)$ is actually the only such pair and, hence, Theorem 1 is unimprovable within the class $\mathscr{A}(n,1,r')$.

First, we shall establish conditions under which a graph in the class mentioned above contains a triangle. Let $x\leqslant a/2$ and suppose that the graph contains a triangle $\Delta(\mathbf{x},\mathbf{y},\mathbf{z})$ with vertices $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$. Denote by $A_{\mathbf{x}}$, $A_{\mathbf{y}}$, $A_{\mathbf{z}}$ the sets of nonzero coordinates of the vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$. All these sets are subsets in $\mathscr{R}_n=\{1,\dots,n\}$. Their pairwise intersections are of cardinality $xn$, and under such conditions they take up minimum space in $\mathscr{R}_n$ (that is, their union has the minimum cardinality) as long as, without loss of generality,

$$\begin{aligned}A_{\mathbf{x}}=\{1,\dots,an\},\qquad A_{\mathbf{y}}&=\{an-xn+1,\dots,an,an+1,\dots,2an-xn\},\\ A_{\mathbf{z}}=\{1,\dots,xn\}&\sqcup\{2an-2xn+1,\dots,2an-xn\}\\ &\sqcup\{2an-xn+1,\dots,3an-3xn\}.\end{aligned}$$

Such an arrangement is possible due to the inequality $x\leqslant a/2$. Consequently, the graph contains a triangle if and only if $3an-3xn\leqslant n$, that is, $a-x\leqslant 1/3$.

Now suppose that $x>a/2$. Then the optimal arrangement of the sets $A_{\mathbf{x}}$, $A_{\mathbf{y}}$, $A_{\mathbf{z}}$ is as follows:

$$A_{\mathbf{x}}=\{1,\dots,an\},\quad A_{\mathbf{y}}=\{an-xn+1,\dots,an,an+1,\dots,2an-xn\},$$

$$A_{\mathbf{z}}=\{1,\dots,xn\}\sqcup\{a+1,\dots,2an-xn\}.$$

It is seen that in this case it is necessary and sufficient that $2a-x\leqslant 1$.

Thus, we have to find out for what values of $a$, $x$ (including the values $a=1/3$ and $x=0$ already known) expression (3.2) attains its minimum value with due regard to the conditions established. Differentiating this expression (to be more exact, the squared expression) with respect to $a$ gives

$$\frac{\partial}{\partial a}((r')^2)=\frac{\partial}{\partial a}\left(\frac{a(1-a)}{2a-2x}\right)=-\frac{a^2-2ax+x}{2(a-x)^2}.$$

Since $a\in[0,1]$, $x\in[0,a]$, we have $x\geqslant ax$ and $a^2\geqslant ax$. Hence, $a^2+x\geqslant 2ax$, which means that the derivative is nonpositive for any $x$. Thus, to minimize (3.2), we should take the value of $a$ as large as possible.

Let $x\leqslant a/2$. Then the largest value of $a$ is equal to $1/3+x$. In this case, $x\leqslant 1/6+x/2$, and hence $x\leqslant 1/3$.

Let $x > a/2$. Then the largest value of $a$ equals $(x+1)/2$. In this case, $x > (x+1)/4$, and so $x > 1/3$.

In other words, for $x \leqslant 1/3$ we have $a = 1/3 + x$, and for $x > 1/3$ we have $a = (x+1)/2$. Let us substitute these values of $a$ into (3.2).

In the first case we obtain the expression $(x+1/3)(2/3-x)/(2/3)$. This is a parabola which opens downward and has its maximum at $x = 1/6$. Therefore, the minimum value over $x \leqslant 1/3$ is attained both at $x = 0$ and at $x = 1/3$ (because of the symmetry of the curve). If $x = 0$, then $a = 1/3$, that is, we arrive at the parameters mentioned in the proof of Theorem 1. If $x = 1/3$, then $a = 2/3$. It is easy to see that this case is obtained from the previous case by replacing nonzero elements by zeros and vice versa at each vertex of the graph. Of course, in the framework of this situation, the exact analogue of Theorem 1 is valid. However, we note that the 'dual change' under which nonzero and zero elements interchange proves to be very useful below, since under such transformation the result remains unchanged.

In the second case we obtain the expression

$$\frac{((x+1)/2)\cdot((1-x)/2)}{1-x} = \frac{x+1}{4}.$$

Of course, it attains its minimum at $x = 1/3$. But then $a = 2/3$, and this case has already been considered.

As a consequence, we see that Theorem 1 is indeed unimprovable.

3.2.4. *On chromatic numbers.* In the previous subsubsections we made no mention of the chromatic numbers of the graphs constructed. They exhibit very interesting behaviour. Indeed, for $a = 1/3$, $x = 0$ we have a classical *Kneser graph* $\mathrm{KG}_{n,an}$ (see [21]). It is well known that $\chi(\mathrm{KG}_{n,k}) = n - 2k + 2$. Thus, the chromatic number of our graph tends to infinity, but the rate of its growth is linear rather than exponential. This means that, as a consequence of Theorem 1, we obtain the estimates

$$\widehat{\varkappa}^{\chi}_{\text{clique}}(3) \geqslant \widehat{\varkappa}^{\chi}_{\text{clique}}(3,1) \geqslant 1.726\dots .$$

However, there are no estimates yet for the other quantities.

We know from § 3.2.3 that in fact there are no other distance graphs on a sphere of the smallest radius. Hence, $\{0,1\}$-points cannot be used to estimate the quantities of the type $\widehat{\varkappa}^{\chi-\exp}_{\text{clique}}(3)$. This is of independent interest, since it forces us to work with points of more complicated structure.

One could attempt to construct graphs on spheres whose radii tend asymptotically to $1/\sqrt{3}$. In this case it can be assumed that $a \sim 1/3$, $x \sim 0$. Unfortunately, even in such a situation chromatic numbers grow subexponentially. This fact follows from the results of [17]. Moreover, it is interesting to see what happens in a similar situation in § 4.

**3.3. Cliques with an arbitrary *k*.** In the previous subsection we discussed the case $k = 3$ in full detail. Now we are ready to consider an arbitrary $k \geqslant 3$. While earlier we first presented a construction and then verified that it is unimprovable, from now on, the line of our reasoning will go backwards. The reason for this is that for $k \geqslant 4$ there are several different constructions even with due account taken of duality. In § 3.3.1 we derive some necessary conditions for the existence of a $k$-clique in a graph of the class $\mathscr{A}(n,1,r_{\text{clique}}(k))$. In § 3.3.2 we establish the sufficiency of these conditions. In § 3.3.3 we prove Theorem 2, which is the analogue of Theorem 1 for arbitrary $k$. In this theorem we estimate the quantities under investigation with the use of various constructions presented in §§ 3.3.1, 3.3.2. In § 3.3.4 we choose the optimal bound among the estimates established in Theorem 2 and evaluate it for small values of $k$. Finally, § 3.3.5 is devoted to chromatic numbers and the corresponding extremal characteristics.

3.3.1. *Necessary conditions for the presence of a $k$-clique.* We are interested in the graph class $\mathscr{A}(n,1,r_{\text{clique}}(k))$. Of course, all of its elements $G_n$ are obtained by normalization of graphs $H_n \in \mathscr{A}(n,1)$. We shall assume, as in § 3.2, that the graphs $H_n$ are determined by parameters $a \in [0,1]$ and $x \in [0,a]$. The following assertion holds.

**Proposition 1.** *Let $k \geqslant 3$ and suppose that a graph $G_n \in \mathscr{A}(n,1,r_{\text{clique}}(k))$ is obtained by normalization of a graph $H_n \in \mathscr{A}(n,1)$ determined by parameters $a$ and $x$, and at the same time $\omega(G_n)=\omega(H_n)=k$, which is to say that the graphs $G_n$ and $H_n$ contain $k$-cliques. Then the value of the parameter $a$ must have the form $a=i/k$ for some (arbitrary) $i\in\{1,\dots,k-1\}$, and the corresponding value of the parameter $x$ must be equal to*

$$\frac{i-1}{k-1}a=\frac{i(i-1)}{k(k-1)}.$$

Note that for $k=3$ we obtain exactly the result of § 3.2.3. Also, note that graphs with $a=i/k$ are dual to graphs with $a=(k-i)/k$. However, for $k\geqslant 4$ there are at least two different types of graphs; to be more exact, the number of types is $[k/2]$.

*Proof of Proposition* 1. First we shall make maximum use of the fact that $H_n$ (this particular graph $H_n$) is determined by the parameters $a$ and $x$ and contains a $k$-clique $K_k$. For this purpose we introduce some additional concepts and notation. Let $F\subseteq H_n$ be an arbitrary subgraph. Each of its vertices is a point with coordinates 0 and 1. There is a total of $n$ coordinates. By the *degree* of the $j$th coordinate in the graph $F$ we mean the number of the graph vertices which have this coordinate equal to 1. This term is quite natural, since the sets of unit components of the vertices of our graph form a hypergraph in which the degree of a coordinate is just the degree of the vertex. We denote the degree by $\deg_F j$.

Consider a clique $K_k\subset H_n$. Each of its vertices has $an$ coordinates 1. Hence, the total number of unit coordinates at its vertices is $kan$. On the other hand, let us set $d_j=\deg_{K_k} j$, $j=1,\dots,n$. Then the total number of unit coordinates can be expressed in another way: $\sum_{j=1}^n d_j$. Thus, we have the equality

$$\sum_{j=1}^{n} d_j = kan.$$

Further, let us make use of the fact that the inner product of any two vectors that correspond to vertices of the clique $K_k$ equals $xn$. Let us sum up all the inner products (of unordered pairs of vectors). On the one hand, we obtain $C_k^2 xn$. On the other hand, it is easily seen that the total sum is $\sum_{j=1}^{n} C_{d_j}^2$. Note that this line of reasoning is convenient when the vertices of the clique are written as rows of a matrix $\mathscr{M}_k$ with entries equal to 0 and 1: this matrix contains $k$ rows and $n$ columns.

As a result, we have a system of equations

$$\begin{cases} \displaystyle\sum_{j=1}^{n} d_j = kan, \\ \displaystyle\sum_{j=1}^{n} C_{d_j}^2 = C_k^2 xn. \end{cases} \tag{3.3}$$

Now we shall take due account of the fact that $G_n$ (this particular graph $G_n$) lies on a sphere of radius $r_{\text{clique}}(k)$, that is, on a sphere of smallest radius. This radius is expressed in terms of $a$, $x$ by formula (3.2), and for a fixed value of $a$, the smaller the value of $x$ the smaller the radius. Thus, let $a$ be fixed. Then it is seen from (3.3) that the sum $\sum_{j=1}^{n} d_j$ is fixed as well. It is well known that under this condition the sum $\sum_{j=1}^{n} C_{d_j}^2$ attains its minimum value (and, in view of (3.3), so does $x$) as long as all quantities $d_j$ are equal to the same quantity $d$. Moreover, if even one equality fails, then the minimum value of $nC_d^2$ cannot be attained.

Thus, we are forced to assume that $d_1 = \cdots = d_n = d \in \{1, \dots, k-1\}$. Then by (3.3) we have $dn = kan$, which yields $d = ka$, and therefore $a = d/k$, as claimed in the proposition (with the only difference that here $i$ is replaced with $d$). At the same time

$$x = \frac{C_d^2}{C_k^2} = \frac{d(d-1)}{k(k-1)}.$$

We note that substituting the values obtained into (3.2) gives exactly $r_{\text{clique}}(k)$. This completes the proof of Proposition 1.

3.3.2. *Sufficient conditions for the presence of a $k$-clique.* It turns out that the necessary conditions derived in the previous subsubsection become sufficient under one additional assumption.

**Proposition 2.** *Let $k \geqslant 3$, $i \in \{1, \dots, k-1\}$, and suppose that a graph $G_n \in \mathscr{A}(n, 1, r_{\text{clique}}(k))$ is obtained by normalizing a graph $H_n \in \mathscr{A}(n, 1)$ specified by the parameters $a = i/k$ and $x = i(i-1)/(k(k-1))$. Let $C_k^i$ divide $n$. Then $\omega(G_n) = \omega(H_n) = k$.*

The additional assumption mentioned above consists of the fact that the dimension is divisible by $C_k^i$.

*Proof of Proposition* 2. We have to prove that the graph $H_n$ contains a $k$-clique. Let us construct this clique explicitly in the form of a matrix $\mathscr{M}_k$ (see § 3.3.1). The matrix will be composed of $n/C_k^i$ identical successive blocks of size $k \times C_k^i$. The columns of each block will represent all possible $k$-tuples containing $i$ 1s.

Let us find the number of 1s at each vertex of the constructed graph. In each block there are as many 1s as there are $i$-element subsets containing a fixed element in a $k$-element set, that is, $C_{k-1}^{i-1}$. Hence, there is a total of

$$C_{k-1}^{i-1} \frac{n}{C_k^i} = \frac{i}{k} n = an$$

1s, which is all right.

Let us calculate the inner product of the vectors corresponding to any two vertices. Within each block, the number of 1s they have in common is equal to the number of $i$-element subsets containing two given elements in a $k$-element set, that is, $C_{k-2}^{i-2}$. In total, we have

$$C_{k-2}^{i-2}\frac{n}{C_k^i}=\frac{i(i-1)}{k(k-1)}n=xn.$$

Again, everything is all right.

The proof of the proposition is complete.

3.3.3. *Estimating the number of $k$-cliques.* The following assertion holds.

**Theorem 2.** *Let $k\geqslant 3$, $i\in\{1,\dots,k-1\}$. Then*

$$\widehat{\kappa}_{\text{clique}}(k)\geqslant\widehat{\kappa}_{\text{clique}}(k,1)\geqslant\log_{\frac{k}{i^{i/k}(k-i)^{(k-i)/k}}}C_k^i.$$

Note that Theorem 1 is a particular case of Theorem 2 for $k=3$, $i=1$.

*Proof of Theorem* 2. Given $k$ and $i$, let us consider only integers $n\in\mathbb{N}$ that are divisible by $C_k^i$ and satisfy the conditions $(i/k)n\in\mathbb{N}$ and

$$\frac{i(i-1)}{k(k-1)}n\in\mathbb{N}.$$

Then Proposition 2 provides an explicit construction for a $k$-clique in the graph $G_n$ which is obtained from a graph $H_n$ with parameters $a=\frac{i}{k}$ and $x=\frac{i(i-1)}{k(k-1)}$. At the same time, Proposition 1 suggests that there exists no better (different) construction. The explicit construction mentioned above can be obtained in many different ways by rearranging columns in the matrix $\mathscr{M}_k$. The number of such rearrangements (and hence the number of $k$-cliques) is

$$X_k(G_n)=\frac{n!}{((n/C_k^i)!)^{C_k^i}}=(C_k^i+o(1))^n.$$

At the same time,

$$\begin{aligned}N(G_n)=C_n^{an}&=\left(\frac{1}{(i/k)^{i/k}((k-i)/k)^{(k-i)/k}}+o(1)\right)^n\\&=\left(\frac{k}{i^{i/k}(k-i)^{(k-i)/k}}+o(1)\right)^n.\end{aligned}$$

Therefore, we in fact have

$$\widehat{\kappa}_{\text{clique}}(k)\geqslant\widehat{\kappa}_{\text{clique}}(k,1)\geqslant\log_{\frac{k}{i^{i/k}(k-i)^{(k-i)/k}}}C_k^i.$$

The proof of the theorem is complete.

3.3.4. *Optimization in Theorem 2*. First of all, let us establish the following result.

**Theorem 3.** *The following inequality holds*:

$$\widehat{\kappa}_{\text{clique}}(2k) \geqslant \widehat{\kappa}_{\text{clique}}(2k,1) \geqslant 2k - \frac{1}{2}\log_2(\pi k) + o(1).$$

*Proof.* In Theorem 2 let us take $2k$ instead of $k$ and set $i$ equal to $k$. Then

$$\frac{2k}{k^{k/(2k)}(2k-k)^{(2k-k)/(2k)}} = 2, \qquad C_{2k}^{k} \sim \frac{2^{2k}}{\sqrt{\pi k}}.$$

It is evident that

$$\log_2\biggl((1+o(1))\frac{2^{2k}}{\sqrt{\pi k}}\biggr) = 2k - \frac{1}{2}\log_2(\pi k) + o(1).$$

The proof of the theorem is complete.

It is clear that for odd values of arguments, the quantities $\widehat{\kappa}_{\text{clique}}(k)$, $\widehat{\kappa}_{\text{clique}}(k,1)$ satisfy almost the same estimate. And this bound is quite remarkable: it turns out that, as $k$ increases,

$$k + O(\log k) \leqslant \widehat{\kappa}_{\text{clique}}(k) \leqslant k,$$

which means that $\widehat{\kappa}_{\text{clique}}(k) \sim k$ with a sharp asymptotic remainder estimate. Below we shall further improve this estimate by considering $m \geqslant 2$ (that is $\{-1,0,1\}$-points and so on).

For small values of $k$ one can draw a table of evaluations of the estimate derived in Theorem 2 (see Table 1).

Table 1

| $i \setminus k$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| 1 | **1.7261** | 2.4652 | 3.2162 | 3.97670 | 4.7447 | 5.5191 | 6.2988 | 7.0830 |
| 2 | **1.7261** | **2.5849** | **3.4214** | 4.2545 | 5.0888 | 5.9256 | 6.7651 | 7.6072 |
| 3 | – | 2.4652 | **3.4214** | **4.3219** | **5.2061** | 6.0846 | 6.9610 | 7.8372 |
| 4 | – | – | 3.2162 | 4.2545 | **5.2061** | **6.1292** | **7.0401** | 7.9450 |
| 5 | – | – | – | 3.9767 | 5.0888 | 6.0846 | **7.0401** | **7.9772** |
| 6 | – | – | – | – | 4.7447 | 5.9256 | 6.9610 | 7.9450 |
| 7 | – | – | – | – | – | 5.5191 | 6.7651 | 7.8372 |
| 8 | – | – | – | – | – | – | 6.2988 | 7.6072 |
| 9 | – | – | – | – | – | – | – | 7.0830 |

It is seen that each time the maximum value (highlighted in bold) is attained for $i=[k/2]$ and for $i=k-[k/2]$. One can show formally that this is indeed the case. Thus, it can be said that Theorem 3 is the strongest corollary of Theorem 2. We shall not go into details here since it is actually an exercise in calculus (though not that easy).

3.3.5. *On chromatic numbers.* Here the situation is much more interesting than in the same-titled subsubsection of the previous subsection. Namely, chromatic numbers become exponential for $k \geqslant 4$. In order to understand this and to perform an appropriate optimization, we must take a certain path.

First of all, we get rid of the constructions which repeat due to duality by letting $i \leqslant [k/2]$. Then, for $i = 1$ we again have a Kneser graph, whose chromatic number is linear (see § 3.2.4). Thus, we fix $k \geqslant 4$ and $i \in \{2, \dots, [k/2]\}$. It is clear that in this case $a \leqslant 1/2$ and $x < a/2$.

Now suppose (for a while) that $an - xn$ is a power of a prime $p$, which means that the dimension $n$ is such that not only $an, xn \in \mathbb{N}$, but also $an - xn = p^{\alpha}$, $\alpha \geqslant 1$. Then the following assertion holds.

**Proposition 3.** *If the values of the parameters $a$ and $x$, as well as the dimension $n$, satisfy all the conditions mentioned above, then the corresponding graph $H_n \in \mathscr{A}(n,1)$ (and hence the graph $G_n \in \mathscr{A}(n,1,r_{\text{clique}}(k))$ as well) has a chromatic number $\chi(H_n)$ bounded below by the quantity*

$$\frac{C_n^{an}}{\sum_{j=0}^{p^{\alpha}-1} C_n^j}.$$

The proof of Proposition 3 is obtained by applying the standard linear algebra method in combinatorics (see [8]). Below we give only a sketch of this proof; all details can be easily reconstructed with the help of the monograph [8], in which very similar assertions are proved.

*Proof of Proposition* 3 (sketch). For the sake of simplicity let us assume that $\alpha = 1$. With each vertex $\mathbf{x} = (x_1, \dots, x_n)$ of the graph $H_n$ we associate a polynomial $F_{\mathbf{x}} \in \mathbb{Z}_p[y_1, \dots, y_n]$ defined by the relations

$$F_{\mathbf{x}}(\mathbf{y}) = \prod_{j \in J} (j - (\mathbf{x}, \mathbf{y})), \qquad J = \{1, \dots, p\} \setminus \{an \mod p\},$$
$$\mathbf{y} = (y_1, \dots, y_n), \qquad y_t^2 = y_t, \qquad t \in \{1, \dots, n\}.$$

The degree of this polynomial is $p - 1$, and by virtue of the relations $y_t^2 = y_t$, its degree in each of the variables is at most 1. Thus, all the polynomials $F_{\mathbf{x}}$ lie in a space of dimension $\sum_{j=0}^{p-1} C_n^j$.

If $W = \{\mathbf{x}_1, \dots, \mathbf{x}_s\}$ is an *independent set* of vertices of the graph $H_n$, which means that it contains no edges or, what is the same, $(\mathbf{x}_r, \mathbf{x}_t) \neq xn$, then taking into account the inequality $x < a/2$ (which yields $an - 2p = an - 2(an - xn) = 2xn - an < 0$) we have $(\mathbf{x}_r, \mathbf{x}_t) \not\equiv xn \pmod p$ for $r \neq t$. Thus, it is easily shown that the polynomials $F_{\mathbf{x}_1}, \dots, F_{\mathbf{x}_s}$ are linearly independent over $\mathbb{Z}_p$ and hence $s \leqslant \sum_{j=0}^{p-1} C_n^j$. As a result, the maximum size $\alpha(H_n)$ of an independent set in $H_n$ does not exceed the same value and, consequently,

$$\chi(H_n) \geqslant \frac{|V(H_n)|}{\alpha(H_n)} \geqslant \frac{C_n^{an}}{\sum_{j=0}^{p-1} C_n^j}.$$

The proof of Proposition 3 is complete.

Modifying the estimate established in Proposition 3 with the use of Stirling's formula we obtain the inequality

$$\chi(H_n) \geqslant \left(\frac{(a-x)^{a-x}(1-a+x)^{1-a+x}}{a^a(1-a)^{1-a}} + o(1)\right)^n. \tag{3.4}$$

All is well, but there is a problem which consists of the fact that the quantity $an-xn$ should hardly ever be considered as exactly equal to $p^\alpha$. At the same time, primality is very important for the proof of Proposition 3, since in the case when $an - xn = q \neq p^\alpha$ the existing estimates are much worse (see [28]).

To come out of these difficulties with honour we should replace the condition $G_n \in \mathscr{A}(n, 1, r_{\text{clique}}(k))$ involved in the definition of the quantities

$$\widehat{\kappa}^{\chi}_{\text{clique}}(k,m), \qquad \widehat{\kappa}^{\chi-\exp}_{\text{clique}}(k,m), \qquad \widehat{\chi}^{\kappa}_{\text{clique}}(k,m)$$

with the condition $G_n \in \mathscr{A}(n,1,r)$, where $r \sim r_{\text{clique}}(k)$ as $n \to \infty$. Of course, that a graph can be drawn on a sphere of the asymptotically smallest radius is hardly more apparent than that it can be drawn on a sphere of the strictly smallest radius. For this reason let us introduce the corresponding quantities

$$\widetilde{\kappa}^{\chi}_{\text{clique}}(k,m), \qquad \widetilde{\kappa}^{\chi-\exp}_{\text{clique}}(k,m), \qquad \widetilde{\chi}^{\kappa}_{\text{clique}}(k,m).$$

With this notation, things are much better. Namely, we take the least prime $p$ such that $an-p < xn$. Then the known results of analytic number theory (see [29]) suggest that $p \sim an - xn$; this means that (3.4) remains valid with the only change in the term $o(1)$, which is inessential for us (here in the definition of the graph $H_n$ we replace the forbidden inner product $xn$ with the quantity $x_1 = an - p \sim xn$).

Further, for each $i \in \{2, \dots, [k/2]\}$, we set $a = i/k$ and $x = i(i-1)/(k(k-1))$. As a result, we arrive at the following theorem.

**Theorem 4.** *The following inequality holds*:

$$\widetilde{\chi}^{\kappa}_{\text{clique}}(k) \geqslant \widetilde{\chi}^{\kappa}_{\text{clique}}(k,1) \geqslant \max_{i\in\{2,\dots,k/2\}} \frac{(a-x)^{a-x}(1-a+x)^{1-a+x}}{a^a(1-a)^{1-a}}.$$

It is easily seen that the maximum value is attained at $i = [k/2]$. And we arrive at a remarkable result: for the same value of $i$ both the number of cliques and the chromatic number attain their maxima. Accordingly, there is no point in looking for optimal estimates for the quantities

$$\widetilde{\kappa}^{\chi}_{\text{clique}}(k,m), \qquad \widetilde{\kappa}^{\chi-\exp}_{\text{clique}}(k,m)$$

separately. They are immediately obtained from the inequalities for $\widetilde{\chi}^{\kappa}_{\text{clique}}(k)$ and $\widehat{\kappa}_{\text{clique}}(k)$.

In Table 2 we present estimates for the quantity $\widetilde{\chi}^{\kappa}_{\text{clique}}(k)$ for particular values of $k$ and $i$.

It is apparent from Table 2 that $i = [k/2]$ is the point of maximum.

Note that

$$\lim_{k\to\infty} \max_{i\in\{2,\dots,[k/2]\}} \frac{(a-x)^{a-x}(1-a+x)^{1-a+x}}{a^a(1-a)^{1-a}} = \frac{1}{2}3^{3/4} = 1.139\dots.$$

Table 2

| $i \setminus k$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| 2 | – | **1.0582** | **1.0641** | 1.0582 | 1.0506 | 1.0436 | 1.0377 | 1.0329 |
| 3 | – | – | – | **1.0857** | **1.0883** | 1.0837 | 1.0769 | 1.0699 |
| 4 | – | – | – | – | – | **1.0995** | **1.1008** | 1.0975 |
| 5 | – | – | – | – | – | – | – | **1.1077** |

Note also that if, in the notation of the quantities considered in this subsubsection, the hat is not yet changed to tilde, then some estimates can still be derived. In other words, graphs with exponentially large chromatic numbers also exist on spheres of exactly (rather than asymptotically) smallest radius. This fact can be derived from the results of [28]. However, in this case there is no elegant optimization and we omit the details to avoid cumbersomeness in our presentation.

In subsequent sections we further improve the estimates obtained. And in the next subsubsection we accurately summarize all the best results of the present section in order to make it easier to compare them in the sequel.

**3.4. Results of the section.** Let us briefly summarize the main results obtained in this section.

1. Theorem 2 (see § 3.3.3) is established, which states that

$$\widehat{\kappa}_{\text{clique}}(k) \geqslant \widehat{\kappa}_{\text{clique}}(k,1) \geqslant \max_{i\in\{1,\dots,k-1\}} \log_{\frac{k}{i^{i/k}(k-i)^{(k-i)/k}}} C_k^i.$$

   In the framework of the $\{0,1\}$-case this theorem is optimal (§§ 3.3.1, 3.3.2).

2. It has been established that in item 1 above the maximum value is attained for $i=[k/2]$. On the one hand, we present Table 1, which contains evaluations of this estimate for $k \leqslant 10$ (see § 3.3.4). On the other hand, we establish Theorem 3 (§ 3.3.4), which states that

$$\widehat{\kappa}_{\text{clique}}(2k) \geqslant \widehat{\kappa}_{\text{clique}}(2k,1) \geqslant 2k - \frac{1}{2}\log_2(\pi k) + o(1).$$

   Although the estimate is very close to the highest possible one and the corresponding evaluations in the table are rather high, these results are subject to further improvement.

3. It is shown that in the framework of the $\{0,1\}$-case the quantity $\widehat{\kappa}^{\chi}_{\text{clique}}(3)$ obeys the same bound as the quantity $\widehat{\kappa}_{\text{clique}}(3)$ (see item 1 above and § 3.2.4). However, the quantities $\widehat{\kappa}^{\chi-\exp}_{\text{clique}}(3)$ and $\widehat{\chi}^{\kappa}_{\text{clique}}(3)$ cannot be estimated in the framework of the same case; moreover, neither can their analogues 'with tildes' (see §§ 3.2.4, 3.3.5). This is an essential drawback observed in the $\{0,1\}$-case.

4. Reasons have been given for replacing the quantities

$$\widehat{\kappa}^{\chi}_{\text{clique}}(k,m), \qquad \widehat{\kappa}^{\chi-\exp}_{\text{clique}}(k,m), \qquad \widehat{\chi}^{\kappa}_{\text{clique}}(k,m)$$

with the quantities

$$\widetilde{\kappa}^{\chi}_{\text{clique}}(k,m), \qquad \widetilde{\kappa}^{\chi-\exp}_{\text{clique}}(k,m), \qquad \widetilde{\chi}^{\kappa}_{\text{clique}}(k,m),$$

which satisfy some nontrivial estimates for $k \geqslant 4$. In particular, Theorem 4 (see § 3.3.5) states that

$$\widetilde{\chi}^{\kappa}_{\text{clique}}(k) \geqslant \widetilde{\chi}^{\kappa}_{\text{clique}}(k,1) \geqslant \max_{i\in\{2,\dots,[k/2]\}} \frac{(a-x)^{a-x}(1-a+x)^{1-a+x}}{a^a(1-a)^{1-a}}.$$

5. It has been established that in item 4 above the maximum value is attained for $i=[k/2]$. A table of evaluations of this estimate for $k \leqslant 10$ is presented (see § 3.3.5). The limit of this estimate as $k \to \infty$ has been obtained. All these results can be improved.

As a result, the main objects of further investigation are the quantities $\widehat{\kappa}_{\text{clique}}(k)$ (whose explicit and asymptotic estimates are of interest) and $\widetilde{\chi}^{\kappa}_{\text{clique}}(k)$ (the case $k=3$ is of particular importance, and the explicit and asymptotic estimates for any $k$ are interesting as well).

## § 4. Constructions with $\{-1,0,1\}$-points

This section is organized in exactly the same way as the previous section. To obtain the desired estimates we shall employ graphs belonging to the class $\mathscr{A}'(n,2)$. In § 4.1 we describe the general properties of such graphs. In § 4.2 we investigate triangles. Subsection 4.3 is devoted to arbitrary $k$-cliques. In § 4.4 we summarize the results of this section together with those of § 3 (see § 3.4).

**4.1. General properties of graphs in the class $\mathscr{A}'(n,2)$.** If a graph $G$ belongs to the class $\mathscr{A}'(n,2)$, then each of its vertices has $l_1$ coordinates with the value of 1, $l_{-1}$ coordinates with the value of $-1$, and $l_0=n-l_1-l_{-1}$ zero coordinates. In § 3.1 and in § 3.2.2 we saw that we lose almost nothing if we write the quantities like $l_i$ in the form of the product of some (rational) constant and the dimension $n$ and assume that such product is a positive integer. In this case everything becomes much easier from the technical viewpoint. It is clear that in this section we can introduce similar notation as well. We set $l_1=an$, $l_{-1}=bn$, and assume without loss of generality that $a\in(0,1)$ and $0<b\leqslant a$. Moreover, $a+b<1$. The cases $a=0$, $b=0$, and $a+b=1$ are not considered here since they have already been examined in the previous section.

Further, the edges may still be specified using the forbidden inner product. For reasons of convenience we shall represent this value in the form $-xn$.

As in § 3, we shall finally consider graphs which are *specified by parameters* $a$, $b$ and $x$. It is evident that these graphs lie on a sphere centred at the point $(a-b,\dots,a-b)$. The squared radius of this sphere equals

$$an(1-a+b)^2+bn(-1-a+b)^2+(1-a-b)n(a-b)^2=n(a+b-(a-b)^2).$$

At the same time the squared length of the edge of our graph can be expressed as $d^2 = 2|\mathbf{v}|^2 - 2(\mathbf{u}, \mathbf{v})$, where the vertices $\mathbf{u}$ and $\mathbf{v}$ form an edge. We have $d^2 = 2n(a+b+x)$. Thus, after normalization we obtain a graph $H \in \mathscr{A}'(n, 2, r')$, where

$$r' = \sqrt{\frac{a+b-(a-b)^2}{2(a+b+x)}}. \tag{4.1}$$

**4.2. Triangles.** This subsection is divided into three parts. In § 4.2.1 we give an example of construction which allows the derivation of new estimates for the quantities $\widehat{\kappa}_{\mathrm{clique}}(3,2)$, $\widehat{\kappa}_{\mathrm{clique}}(3)$. In § 4.2.2 we show that the construction presented in § 4.2.1 is optimal. Subsection 4.2.3 is devoted to chromatic numbers. In particular, we shall see that the case of $\{-1,0,1\}$-points is radically better than the one considered earlier: in the framework of this case exponential estimates for the chromatic number can be derived, and hence it is reasonable to look for optimal estimates for the quantities $\widehat{\kappa}^{\chi-\exp}_{\mathrm{clique}}(3)$ and $\widetilde{\chi}^{\kappa}_{\mathrm{clique}}(3)$.

4.2.1. *The case $a = b = x$.* In this case $r' = 1/\sqrt{3} = r_{\mathrm{clique}}(3)$, as required. It is easily seen that for all $a \leqslant 1/3$ the corresponding graph $G_n$ in the class $\mathscr{A}'(n, 2, r')$ contains triangles. The most frequently occurring construction of a triangle (before normalization) has the form

$$\begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0
\end{array}$$

Here $a = 1/4$. Clearly, the number of such constructions for $a < 1/3$ is

$$C_n^{an} C_{n-an}^{an} (C_{an}^{an/2})^2 C_{n-2an}^{an} C_{an}^{an/2} = \left(\frac{2^{3a}}{(1-3a)^{1-3a} a^{3a}} + o(1)\right)^n$$

(we omit the integer parts, since after all they affect only the term $o(1)$ in the base of the power expression). Thus,

$$X_3(G_n) \geqslant \left(\frac{2^{3a}}{(1-3a)^{1-3a} a^{3a}} + o(1)\right)^n.$$

It can be demonstrated that this inequality is unimprovable (only $o(1)$ can be changed), but we shall not go into details here (or rather we refer the reader to §§ 4.3.2, 4.3.3).

Further,

$$N(G_n) = \left(\frac{1}{(1-2a)^{1-2a} a^{2a}} + o(1)\right)^n.$$

As a result, we obtain

**Theorem 5.** *For any $a < 1/3$ the following estimate is valid*:

$$\widehat{\kappa}_{\mathrm{clique}}(3) \geqslant \widehat{\kappa}_{\mathrm{clique}}(3,2) \geqslant \log_{\frac{1}{(1-2a)^{1-2a} a^{2a}}} \left(\frac{2^{3a}}{(1-3a)^{1-3a} a^{3a}}\right).$$

Taking the numerical maximum with respect to $a$ in Theorem 5, for $a=0.2144\dots$ we obtain the estimate

$$\widehat{\kappa}_{\text{clique}}(3) \geqslant \widehat{\kappa}_{\text{clique}}(3,2) \geqslant 1.8404\dots,$$

and this is much higher than $1.7261\dots$ obtained in Theorem 1.

4.2.2. *The estimate established in Theorem 5 is optimal.* The following proposition is valid.

**Proposition 4.** *Suppose that a graph $G_n \in \mathscr{A}'(n,2,r_{\text{clique}}(3))$ is obtained by normalization of a graph $H_n \in \mathscr{A}'(n,2)$ determined by parameters $a$, $b$ and $x$, and at the same time $\omega(G_n)=\omega(H_n)=3$, that is, the graphs $G_n$ and $H_n$ contain triangles. Without loss of generality it may be supposed that $a \geqslant b$. Then either $a=b=x$, $0<b\leqslant 1/3$, or $a-b=1/3$, $b=x$, $0<b\leqslant 1/3$.*

Proposition 4 will be established later as a consequence of a similar general result for $k$-cliques (see § 4.3).

Proposition 4 suggests that, in addition to the case considered in the previous subsubsection, there is at least one more situation in which the graphs we are interested in lie on spheres of minimum radius and at the same time contain triangles. Thus, to prove the optimality of the estimate in Theorem 5 one should look for constructions with parameters $a-b=1/3$, $b=x$, $0<b\leqslant 1/3$ and, if such constructions exist, make sure that they provide weaker estimates for the quantity $\widehat{\kappa}_{\text{clique}}(3)$.

First of all, let us understand why for the values of the parameters mentioned above we have $r'=1/\sqrt{3}$. As a matter of fact, $a=b+1/3$, whence it follows that

$$\frac{a+b-(a-b)^2}{2(a+b+x)} = \frac{a+b-1/9}{2(a+2b)} = \frac{2b+2/9}{2(3b+1/3)} = \frac{1}{3}.$$

Now let us present a typical construction of a triangle (before normalization):

$$\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0\\
1 & 1 & -1 & -1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0\\
-1 & -1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}$$

Here $a=1/2$ and $b=x=1/6$. In the general case the construction is as follows. First we choose a set $B_1$ consisting of $bn$ coordinate positions for the value of $-1$ at the first vertex of the triangle. Then we fix a set $B_2$ which is disjoint from $B_1$ and consists of $bn$ coordinate positions for the value of $-1$ at the second vertex of the triangle; after that we choose $B_3$ similarly: $B_3\cap B_1\cap B_2=\varnothing$, $|B_3|=bn$, where $B_3$ is the set of coordinate positions for the value of $-1$ at the third vertex of the triangle; then at the $i$th vertex we fill all coordinate positions belonging to the set $(B_1\cup B_2\cup B_3)\setminus B_i$ with 1s, and after that $an-2bn$ 1s are still to be filled in at each vertex; we arrange the corresponding coordinate positions in three mutually disjoint sets of cardinality $(a-2b)n$ embedded in a set of cardinality $n-3bn$ (here, of course, $(3a-6b)n=n-3bn$, as long as we have $a-b=1/3$). As a result, for

$b < 1/3$ we obtain

$$N(G_n) = C_n^{bn} C_{n-bn}^{(b+1/3)n} = \left( \frac{1}{b^b (b+1/3)^{b+1/3} (2/3-2b)^{2/3-2b}} + o(1) \right)^n,$$

$$X_3(G_n) \geqslant C_n^{bn} C_{n-bn}^{bn} C_{n-2bn}^{bn} C_{n-3bn}^{(a-2b)n} C_{(1-a-b)n}^{(a-2b)n} = \left( \frac{1}{b^{3b}(1/3-b)^{1-3b}} + o(1) \right)^n.$$

(It can be demonstrated that the estimate for $X_3$ is asymptotically unimprovable, but we shall not labour the point; rather we refer the reader to §§ 4.3.2 and 4.3.3.)

This immediately yields the estimate

$$\widehat{\kappa}_{\text{clique}}(3) \geqslant \widehat{\kappa}_{\text{clique}}(3,2) \geqslant \log_{\frac{1}{b^b (b+1/3)^{b+1/3} (2/3-2b)^{2/3-2b}}} \left( \frac{1}{b^{3b}(1/3-b)^{1-3b}} \right).$$

Numerical optimization shows that the maximum value is attained for $b \approx 0.056$ and this maximum value is equal to $1.813\dots$. Since $1.813 < 1.84$, this proves the optimality of the estimate obtained in § 4.2.1.

4.2.3. *On chromatic numbers.* As in § 3.3.5, we shall finally employ the linear algebra method. As we now understand, this method is based on the primality of various numbers. For this reason we turn directly to the quantities $\widetilde{\kappa}_{\text{clique}}^{\chi}(3)$, $\widetilde{\kappa}_{\text{clique}}^{\chi-\exp}(3)$ and $\widetilde{\chi}_{\text{clique}}^{\kappa}(3)$, that is, we allow the desired graphs to lie on spheres of asymptotically minimal, rather then strictly minimal radius. First of all we shall investigate the 'kappa-type' quantities. It turns out that in this case the first of these quantities is of no concern at all. The point is that we shall manage to estimate the second quantity exactly as the quantity 'kappa hat' investigated in § 4.2.1. In other words, we shall demonstrate that a slight modification of the graph on which the estimate $\widehat{\kappa}_{\text{clique}}(3) \geqslant 1.8404\dots$ is realized gives a new graph which lies on a sphere of radius $\sim 1/\sqrt{3}$, exhibits the same relationship between the number of vertices and triangles, and has exponentially large chromatic number. Thus, we arrive at

**Theorem 6.** *The following estimate holds*: $\widetilde{\kappa}_{\text{clique}}^{\chi-\exp}(3) \geqslant 1.8404$.

*Proof.* Let $a \approx 0.2144$ be the rational number on which the estimate $\widehat{\kappa}_{\text{clique}}(3) \geqslant 1.8404\dots$ is realized. We assume that an integer $n$ is such that not only is the number $an$ an integer, but in addition, it is an even integer. It is easily seen that, as usual, this assumption does not affect the desired estimates (cf. § 3.2.2). Take the minimum prime $p$ such that $2an - 4p < -an$. As in § 3.3.5, we refer to [29] in asserting that $p \sim 3an/4$. Take $b = a$ and consider the value $y = 2an - 4p \sim -an$ as the forbidden value of inner product. Note that $y = -xn$; however, here $x$ is not a constant as it was everywhere earlier, but only an asymptotic constant: $x \sim a$. It is for this reason that we have changed the notation.

Consider the graphs $H_n \in \mathscr{A}'(n,2)$ determined by the parameters $a$, $b$ and the forbidden inner product $y$, as well as the corresponding graphs $G_n \in \mathscr{A}(n,2,r')$. Since $y \sim -an$, our graphs $G_n$ have the same asymptotic characteristics as the graphs denoted the same in § 4.2.1: they exhibit the same relationship between the number of vertices and triangles and $r' \sim r_{\text{clique}}(3) = 1/\sqrt{3}$. Hence, verifying that $\chi(G_n) = \chi(H_n) \geqslant (c+o(1))^n$, $c > 1$, will complete the proof of the theorem.

We shall employ the estimate $\chi(G) \geqslant |V(G)|/\alpha(G)$ (see § 3.3.5). It is evident that $|V(H_n)| = C_n^{2an} C_{2an}^{an}$. We see that $\alpha(H_n) \leqslant C_n^{2an} \sum_{i=0}^{p-1} C_{2an}^i$. Take $t = C_n^{2an}$

and let $A_1, \dots, A_t$ be all the $2an$-element subsets of the set $\{1, \dots, n\}$. Further, let $V_i \subset V(H_n)$ be the totality of all vector strings whose coordinates containing the values $\pm 1$ comprise the set $A_i$. Certainly, $V(H_n) = V_1 \sqcup \dots \sqcup V_t$ and $\alpha(H_n) \leqslant t\alpha(H_n|_{V_1})$. Introduce the notation $F_n = H_n|_{V_1}$. Without loss of generality it may be assumed that the vertices of the graph $F_n$ correspond to $2an$-dimensional vectors with the same number of coordinates equal to 1 and $-1$. As before, the edges of $F_n$ are determined by the inner products $y$. Let us show that $\alpha(F_n) \leqslant \sum_{i=0}^{p-1} C_{2an}^i$.

First, it is easily seen that for any $\mathbf{x}, \mathbf{y} \in V(F_n)$ we have $(\mathbf{x}, \mathbf{y}) \equiv 0 \pmod 4$. This is because for each vector the numbers of components equal to 1 and to $-1$ are both even and the inner product of any vector with itself, being equal to $2an$, is divisible by 4. Hence, the inner product can be congruent to $2an$ modulo $p$ only when it equals $2an$ (the vectors coincide) or $2an - 4p$ (the vectors form an edge): numbers of the form $2an - p$, $2an - 2p$, and so on, are not divisible by four, and $2an - 8p < -4an$, whereas in our case the least inner product of vertex vectors is $-2an$.

We construct polynomials $P_{\mathbf{x}} \in \mathbb{Z}_p[y_1, \dots, y_{2an}]$ associated with the vertices $\mathbf{x}$ of the graph $F_n$, $\mathbf{x} = (x_1, \dots, x_{2an})$,

$$P_{\mathbf{x}}(\mathbf{y}) = \prod_{j \in J} (j - (\mathbf{x}, \mathbf{y})), \qquad J = \{1, \dots, p\} \setminus \{2an \mod p\},$$
$$\mathbf{y} = (y_1, \dots, y_{2an}), \qquad y_l^2 = 1, \quad l \in \{1, \dots, 2an\}.$$

The degree of each polynomial of this kind is $p - 1$, and owing to the equalities $y_l^2 = 1$, the degree of such a polynomial in each of the variables is at most one. Thus, all polynomials $P_{\mathbf{x}}$ lie in a space of dimension $\sum_{i=0}^{p-1} C_{2an}^i$. We take an arbitrary independent set of vertices $W = \{\mathbf{x}_1, \dots, \mathbf{x}_s\}$ (that is, $(\mathbf{x}_i, \mathbf{x}_j) \neq 2an - 4p$ for $i \neq j$), and see that the polynomials $P_{\mathbf{x}_1}, \dots, P_{\mathbf{x}_s}$ are linearly independent over $\mathbb{Z}_p$. Thus, the required estimate is established.

Finally, we have

$$\chi(H_n) \geqslant \frac{|V(H_n)|}{\alpha(H_n)} \geqslant \frac{C_n^{2an} C_{2an}^{an}}{C_n^{2an} \sum_{i=0}^{p-1} C_{2an}^i} = \frac{C_{2an}^{an}}{\sum_{i=0}^{p-1} C_{2an}^i}$$
$$= \left( \left(\frac{3}{4}\right)^{3a/4} \left(\frac{5}{4}\right)^{5a/4} + o(1) \right)^n > (1.013)^n$$

for large values of $n$, which completes the proof of the theorem.

Now let us consider the quantity $\widetilde{\chi}^{\kappa}_{\mathrm{clique}}(3)$. It is quite easy to establish the following fact.

**Theorem 7.** *The following estimate holds*: $\widetilde{\chi}^{\kappa}_{\mathrm{clique}}(3) \geqslant 1.021$.

*Proof.* In the proof of Theorem 6 we constructed graphs $H_n$ which were uniquely determined by the value of the parameter $a \approx 0.2144$. Actually, one may similarly define a sequence of graphs for $a \in (0, 1/3)$. And all our reasonings apply to this case as well. Namely, for any $a$ in this interval, we have $X_3(H_n) \geqslant N^{\kappa}(H_n)$ with $\kappa > 1$ and

$$\chi(H_n) \geqslant \left( \left(\frac{3}{4}\right)^{3a/4} \left(\frac{5}{4}\right)^{5a/4} + o(1) \right)^n.$$

It is easily seen that the maximum value of the expression $(3/4)^{3a/4}(5/4)^{5a/4}$ is attained at $a = 1/3$, and this maximum value equals $1.021\dots$. Taking $a \in \mathbb{Q}$ close enough to $1/3$ so as to satisfy the inequality $(3/4)^{3a/4}(5/4)^{5a/4} > 1.021$, we guarantee that $H_n$ possesses all the desired properties. The proof of the theorem is complete.

Actually, we have one more situation remaining: the case $a-b = 1/3$. Curiously, in this situation there is nothing to be had: there is no value of $b$ for which the linear algebra method gives any substantial (that is, exponential) estimates. This can be shown in a formal way, but we need hardly go into details here. As a result, we have the inequality established in Theorem 7, and it is the best estimate we have so far.

**4.3. Cliques with an arbitrary value of $k$.** This subsection is organized exactly as the same-titled § 3.3. In § 4.3.1 we present necessary conditions for the existence of a clique in a graph of the class $\mathscr{A}'(n, 2, r_{\text{clique}}(k))$. In § 4.3.2 we show that these conditions are actually sufficient as well. Subsubsections 4.3.3 and 4.3.4 are devoted to deriving estimates for the number of cliques, and in § 4.3.5 we discuss chromatic numbers.

4.3.1. *Necessary conditions for the existence of a $k$-clique.* The following proposition holds.

**Proposition 5.** *Let $k \geqslant 3$. Suppose that the graph $G_n \in \mathscr{A}'(n, 2, r_{\text{clique}}(k))$ is obtained by normalization of a graph $H_n \in \mathscr{A}'(n, 2)$ determined by parameters $a$, $b$ and $x$, and at the same time $\omega(G_n) = \omega(H_n) = k$, which is to say that the graphs $G_n$ and $H_n$ contain $k$-cliques. Without loss of generality it may be assumed that $a \geqslant b$. Then the quantity $a-b$ must have the form $a-b = s/k$ for some (arbitrary) $s \in \{0, \dots, k-2\}$, and the corresponding value of $x$ must be equal to $(a+b-k(a-b)^2)/(k-1)$. At the same time, $2\lceil bk\rceil \leqslant k-s$.*

It is immediately seen why Proposition 4 is a direct corollary of Proposition 5. Indeed, in Proposition 4 we have the equality $k = 3$. By Proposition 5 we have $s \in \{0, 1\}$. If $s = 0$, then $a = b$ and $x = 2a/2 = a = b$. Moreover, $2\lceil 3b\rceil \leqslant 3-0 = 3$, whence it follows that $b \leqslant 1/3$. And if $s = 1$, then $a-b = 1/3$, $x = (2b+1/3-3\cdot 1/9)/2 = b$, and $2\lceil 3b\rceil \leqslant 3-1 = 2$, whence it follows that $b \leqslant 1/3$.

*Proof of Proposition* 5. It is quite natural to expect that here we shall in some way generalize the idea of the proof of Proposition 1. And this is indeed what we shall do.

Consider an arbitrary $k$-clique $K_k \subset H_n$. Its vectors form a matrix (cf. §§ 3.3.1 and 3.3.2), which we denote by $\mathscr{M}_k$: this matrix has $k$ rows and $n$ columns and its entries are the numbers $-1$, 0, 1. Denote by $l_j$, $j \in \{1, \dots, n\}$, the number of $-1$s in the $j$th column and by $s_j$ the difference between the number of 1s and $-1$s in the same column. In each row of the matrix the difference between the number of 1s and $-1$s is $(a-b)n$. Hence, in total, the matrix $\mathscr{M}_k$ contains $k(a-b)n$ more 1s than $-1$s. On the other hand, the same difference may be calculated as the total sum of the differences in each column. As a result, we obtain the equality

$$\sum_{j=1}^{n} s_j = k(a-b)n.$$

Now let us take the sum of all pairwise inner products. Each of them equals $-xn$ since $K_k$ forms a clique in $H_n$. In total, we have $-C_k^2 xn$. On the other hand, we can consider the contribution to the total sum made by each column of the matrix $\mathscr{M}_k$. As we know, the $j$th column contains $l_j$ $-1$s and $l_j + s_j$ 1s. Hence, the contribution of this column is

$$C_{l_j}^2 + C_{l_j+s_j}^2 - l_j(l_j+s_j) = -l_j + \frac{s_j^2 - s_j}{2}.$$

Summing up these quantities, we obtain the equality

$$\sum_{j=1}^{n}\biggl(-l_j + \frac{s_j^2 - s_j}{2}\biggr) = -C_k^2 xn.$$

It is easily seen that $\sum_{j=1}^{n} l_j = kbn$, whence it follows that

$$-kbn + \sum_{j=1}^{n} \frac{s_j^2 - s_j}{2} = -C_k^2 xn$$
$$\iff \quad k(k-1)xn = 2kbn - \sum_{j=1}^{n} s_j(s_j-1) = k(a+b)n - \sum_{j=1}^{n} s_j^2.$$

As a result, we arrive at the system of equations

$$\begin{cases} \displaystyle\sum_{j=1}^{n} s_j = k(a-b)n, \\ \displaystyle k(k-1)xn = k(a+b)n - \sum_{j=1}^{n} s_j^2. \end{cases} \tag{4.2}$$

As in § 3.3.1, we also keep in mind the fact that the graph $G_n$ (this particular graph $G_n$) lies on a sphere of minimum radius. Hence, the value of $x$ (for given values of $a$ and $b$) should be as large as possible (see formula (4.1)). The first equation of system (4.2) suggests that the sum $\sum_{j=1}^{n} s_j$ is fixed. As in § 3.3.1, we immediately see with regard to the second equation of system (4.2) that $x$ attains its largest value if and only if all the $s_j$ are equal to the same $s \in \{0,\dots,k-1\}$. Thus, we necessarily have

$$\begin{cases} s = k(a-b), \\ k(k-1)x = k(a+b) - s^2. \end{cases} \tag{4.3}$$

Therefore, we in fact have

$$a - b = \frac{s}{k}, \qquad x = \frac{a+b-k(a-b)^2}{k-1}.$$

Moreover, substituting this value of $x$ into (4.1), we obtain

$$(r')^2 = \frac{a+b-(a-b)^2}{2(a+b+x)} = \frac{a+b-(a-b)^2}{2(a+b+(a+b)/(k-1)-k(a-b)^2/(k-1))}$$
$$= \frac{a+b-(a-b)^2}{(2k/(k-1))(a+b-(a-b)^2)} = \frac{k-1}{2k} = r^2_{\text{clique}}(k).$$

It remains to demonstrate that $s \leqslant k-2$ and $2\lceil bk\rceil \leqslant k-s$. Assume that $s \geqslant k-1$ and that there is a column of the matrix $\mathscr{M}_k$ which contains at least one entry equal to $-1$. Then the same column contains at least $s+1$ 1s, which means that the total number of 1s and $-1$s in this column is at least $s+2 \geqslant k+1$. This is impossible and thus no column of matrix $\mathscr{M}_k$ contains entries equal to $-1$, which means that there are no $-1$s in the matrix at all, and this is also impossible (we have $b>0$). Now let us establish the second inequality. By the pigeonhole principle the matrix $\mathscr{M}_k$ has a column containing at least $\lceil (bn)k/n\rceil$ entries equal to $-1$. Then, however, the same column contains at least $\lceil (bn)k/n\rceil + s$ 1s. Hence,

$$\left\lceil \frac{(bn)k}{n} \right\rceil + \left\lceil \frac{(bn)k}{n} \right\rceil + s = 2\lceil bk\rceil + s \leqslant k,$$

which was to be shown.

The proof of the proposition is complete.

4.3.2. *Sufficient conditions for the existence of a k-clique.* The following assertion holds.

**Proposition 6.** *Let $k \geqslant 3$, $s \in \{0,\dots,k-2\}$, and let $b$ satisfy $2\lceil bk\rceil \leqslant k-s$. Suppose that the graph $G_n \in \mathscr{A}'(n,2,r_{\text{clique}}(k))$ is obtained by normalization of a graph $H_n \in \mathscr{A}'(n,2)$ determined by the parameters $b$, $a=b+s/k$ and $x=(a+b-k(a-b)^2)/(k-1)$. Consider arbitrary nonnegative rational numbers $n'_j$ satisfying the system of equations*

$$\begin{cases} \displaystyle\sum_{j=0}^{[(k-s)/2]} j n'_j C_k^j C_{k-j}^{j+s} = kb, \\ \displaystyle\sum_{j=0}^{[(k-s)/2]} n'_j C_k^j C_{k-j}^{j+s} = 1, \end{cases} \tag{4.4}$$

*and let $n$ be such that the numbers $n_j = n'_j n$ are integers. Then $\omega(G_n)=\omega(H_n)=k$.*

Proposition 6 has a similar meaning to Proposition 2. In Proposition 2 it was required that $n$ be divisible by some $C_k^i$, and for further reasonings only the fact that $C_k^i$ is a fixed number was important. Here, the conditions of divisibility of $n$ appear more complicated. However, the point is still that the numbers $n'_j$ are enclosed in some fixed limits. The question may arise of their existence. However, it is easily seen that such numbers always exist. Indeed, since $s \leqslant k-2$, the integer $t=[(k-s)/2]$ is nonzero. Let us set the numbers $n'_1,\dots,n'_{t-1}$ equal to zero (of course, if $t \geqslant 2$). Then only $n'_0$ and $n'_t$ will remain in system (4.4). We know that $2\lceil bk\rceil \leqslant k-s$, whence it follows that $bk \leqslant t$ and, therefore, $n'_t C_k^t C_{k-t}^{t+s} \leqslant 1$

(by virtue of the first equation of the system); in this case $n'_t$ is rational and positive. The second equation of the system yields $n'_0 C_k^s = 1 - n'_t C_k^t C_{k-t}^{t+s} \geqslant 0$, and so $n'_0$ is also rational and nonnegative. In general, there are usually many more solutions. Below we shall consider the optimal choice of the parameters which will also include the numbers $n'_j$. Then it will become quite apparent that the problem consists not of finding out whether the system has a solution, but of organizing an accurate search through such solutions in order to find the best one in some sense.

Let us give two more useful examples. First, let $k = 3$, $s = 0$. Then $a = b = x$, which means that we are in the situation of § 4.2.1. System (4.4) assumes the form

$$\begin{cases} 6n'_1 = 3a, \\ n'_0 + 6n'_1 = 1; \end{cases} \tag{4.5}$$

so $n'_0 = 1 - 3a$, $n'_1 = a/2$ and for any $a \leqslant 1/3$ these numbers are well-defined. Moreover, they appear explicitly in § 4.2.1 and this is discussed further below. Second, let $k = 3$, $s = 1$. Then we are in the situation of § 4.2.2, in which we definitely have $n'_0 = 1/3 - b$, $n'_1 = b$. These parameters also appear explicitly in § 4.2.2 and this is also given attention below.

Thus, Proposition 6 states that Proposition 5 is unimprovable in the sense that under the appropriate requirement of divisibility of $n$ the necessary conditions turn out to be sufficient. Let us prove it.

*Proof of Proposition* 6. We shall construct the desired clique in the form of a matrix $\mathscr{M}_k$. Let $n'_0, \dots, n'_t$ be some roots of system (4.4). We shall take only those $n'_{j_1}, \dots, n'_{j_r}$ which are strictly positive and consider the corresponding $n_{j_1}, \dots, n_{j_r}$. Let the first $C_k^{j_1} C_{k-j_1}^{j_1+s}$ columns of the matrix $\mathscr{M}_k$ be all possible vectors having $j_1$ entries equal to $-1$ and $j_1 + s$ entries equal to 1. Duplicating the block defined above $n_{j_1}$ times we form the first part of the matrix $\mathscr{M}_k$ consisting of $n_{j_1} C_k^{j_1} C_{k-j_1}^{j_1+s}$ columns. We continue to the right with a similar portion of $n_{j_2} C_k^{j_2} C_{k-j_2}^{j_2+s}$ columns, which consists of $n_{j_2}$ blocks, each containing all possible vector columns with $j_2$ entries equal to $-1$ and $j_2 + s$ entries equal to 1. Proceeding in this way until the indices $j_1, \dots, j_r$ are exhausted we obtain a matrix containing, exactly as we need, $k$ rows and

$$\sum_{\nu=1}^{r} n_{j_\nu} C_k^{j_\nu} C_{k-j_\nu}^{j_\nu+s} = \sum_{j=0}^{t} n_j C_k^j C_{k-j}^{j+s} = \sum_{j=0}^{t} n'_j n C_k^j C_{k-j}^{j+s} = n \sum_{j=0}^{t} n'_j C_k^j C_{k-j}^{j+s} = n$$

columns.

It remains to verify that each row of the matrix $\mathscr{M}_k$ contains exactly $bn$ entries equal to $-1$ and $an$ entries equal to 1, and that the inner product of each pair of vector rows of this matrix equals $-xn$. It is evident that, by construction, all rows of $\mathscr{M}_k$ contain the same number of entries equal to $-1$ and the same number of entries equal to 1. At the same time the total (over the entire matrix) number of entries equal to $-1$ is $kbn$ in view of the first equation of system (4.4). And the total number of entries equal to 1 is $kbn + sn = kn(b + s/k) = kan$. Thus, it remains to calculate the inner products.

Again, it is clear that all inner products are equal. They are calculated as follows. Without loss of generality we take the first two rows of the matrix $\mathscr{M}_k$ and consider any block of this matrix corresponding to some index $j \in \{j_1, \dots, j_r\}$. Within this block the rows under consideration have the same positions with entries equal to $-1$ in $C_{k-2}^{j-2} C_{k-j}^{j+s}$ cases, the same positions with entries equal to 1 in $C_{k-2}^{j} C_{k-j-2}^{j+s-2}$ cases, and the entry equal to $-1$ is below or above the entry equal to 1 in $2C_{k-2}^{j-1} C_{k-j-1}^{j+s-1}$ cases. As a result, the contribution of this block to the inner product under study is

$$C_{k-2}^{j-2} C_{k-j}^{j+s} - 2C_{k-2}^{j-1} C_{k-j-1}^{j+s-1} + C_{k-2}^{j} C_{k-j-2}^{j+s-2}.$$

Hence, the entire inner product equals

$$\begin{aligned}
&\sum_{j=0}^{t} n_j (C_{k-2}^{j-2} C_{k-j}^{j+s} - 2C_{k-2}^{j-1} C_{k-j-1}^{j+s-1} + C_{k-2}^{j} C_{k-j-2}^{j+s-2}) \\
&\qquad = \sum_{j=0}^{t} n_j C_k^j C_{k-j}^{j+s} \biggl( \frac{j(j-1)}{k(k-1)} - \frac{2j(j+s)}{k(k-1)} + \frac{(j+s)(j+s-1)}{k(k-1)} \biggr) \\
&\qquad = \frac{1}{k(k-1)} \sum_{j=0}^{t} n_j C_k^j C_{k-j}^{j+s} (s^2 - s - 2j) = \frac{(s^2 - s - 2kb)n}{k(k-1)} \\
&\qquad = n \frac{-2b - s/k + s^2/k}{k-1} = n \frac{-a - b + k(a-b)^2}{k-1} = -xn.
\end{aligned}$$

The proof of the proposition is complete.

Let us turn once again to the cases $k = 3$, $s = 0$ (§ 4.2.1) and $k = 3$, $s = 1$ (§ 4.2.2). As we know, in the first of these situations, $n_0' = 1 - 3a$, $n_1' = a/2$. If we look closely at the construction of the triangle in § 4.2.1, then we shall see that it actually represents the matrix $\mathscr{M}_k$ described in the proof of Proposition 6. In this matrix, $r = 2$, $j_1 = 0$, and $j_2 = 1$, which is to say that both the numbers $n_0'$ and $n_1'$ are taken into account. Moreover, $n = 16$, $a = 1/4$, the only vector column containing no entries equal to $-1$ and no entries equal to 1 (since $s = 0$) appears exactly $n_0 = n - 3an = 16 - 12 = 4$ times, six vector columns containing one entry equal to $-1$ and one entry equal to 1 are repeated exactly $n_1 = an/2 = 2$ times each. However, the duplication of the blocks is arranged in a slightly different way, but it only means that the clique is not unique, and this is the subject of discussion in the next subsubsection. The situation in the case of § 4.2.2 is quite similar.

4.3.3. *An estimate for the number of $k$-cliques.* The following assertion holds.

**Theorem 8.** *Let $k \geqslant 3$, $s \in \{0, \dots, k-2\}$, $t = [(k-s)/2]$, and a number $b$ be such that $2\lceil bk \rceil \leqslant k - s$. Consider arbitrary nonnegative rational numbers $n_j'$ satisfying system (4.4) and denote*

$$u = \frac{1}{b^b (b + s/k)^{b+s/k} (1 - 2b - s/k)^{1-2b-s/k}}, \qquad v = \frac{1}{\prod_{j=0}^{t} (n_j')^{n_j' C_k^j C_{k-j}^{j+s}}}.$$

*Here $(n_j')^{n_j'} = 1$ whenever $n_j' = 0$. Then*

$$\widehat{\kappa}_{\text{clique}}(k) \geqslant \widehat{\kappa}_{\text{clique}}(k, 2) \geqslant \log_u v.$$

Let us turn once again to the situations of §§ 4.2.1 and 4.2.2. In the first of these cases we have

$$k=3, \quad s=0, \quad t=1, \quad 0<b<\frac{1}{3}, \quad a=b, \qquad n_0'=1-3a, \quad n_1'=\frac{a}{2},$$

and therefore

$$u=\frac{1}{a^{2a}(1-2a)^{1-2a}}, \qquad v=\frac{1}{(1-3a)^{1-3a}(a/2)^{6\cdot a/2}}=\frac{2^{3a}}{(1-3a)^{1-3a}a^{3a}}.$$

This agrees completely with Theorem 5. And it is now apparent that Theorem 5 is unimprovable.

In the second case we have

$$k=3, \quad s=1, \quad t=1, \quad 0<b<\frac{1}{3}, \quad a=b+\frac{1}{3}, \qquad n_0'=\frac{1}{3}-b, \quad n_1'=b,$$

and therefore

$$u=\frac{1}{b^b(b+1/3)^{b+1/3}(2/3-2b)^{2/3-2b}},$$
$$v=\frac{1}{(1/3-b)^{3(1/3-b)}b^{3b}}=\frac{1}{(1/3-b)^{1-3b}b^{3b}}.$$

Again this agrees completely with the estimate obtained in § 4.2.2, which means that the result obtained in that subsubsection is unimprovable as well.

*Proof of Theorem* 8. Suppose that all parameters in Proposition 6 and Theorem 8 are fixed. Then Proposition 6 provides an explicit construction of a clique. At the same time, Proposition 5 guarantees that there exists no (other) better construction. The explicit construction mentioned above can be obtained in many different ways by rearranging columns in the matrix $\mathscr{M}_k$. The number of such rearrangements (and thus the number of $k$-cliques) is

$$\begin{aligned} X_k(G_n)&=\frac{n!}{(n_0!)^{C_k^s}(n_1!)^{C_k^1C_{k-1}^{s+1}}\dots(n_t!)^{C_k^tC_{k-t}^{t+s}}}\\ &=\left(\frac{1}{\prod_{j=0}^t(n_j')^{n_j'C_k^jC_{k-j}^{j+s}}}+o(1)\right)^n=(v+o(1))^n. \end{aligned}$$

The next-to-last expression is obtained with due account taken of Stirling's formula, and it is correct as long as not all the quantities $n_j'$ are equal to zero or one.

At the same time,

$$N(G_n)=C_n^{bn}C_{n-bn}^{an}=C_n^{bn}C_{n-bn}^{bn+s/k}=(u+o(1))^n.$$

The proof of the theorem is complete.

4.3.4. *Optimization in Theorem 8.* Following the lines of reasoning presented in § 3.3.4, we start by establishing the following theorem.

**Theorem 9.** *The following inequality holds*:

$$\widehat{\kappa}_{\text{clique}}(3k) \geqslant \widehat{\kappa}_{\text{clique}}(3k, 2) \geqslant 3k - \frac{1}{2}\log_3 k + o(\ln k).$$

One can see that Theorem 9 improves the similar Theorem 3, since the logarithm to base three is less than the binary logarithm by a constant factor, and the quantity $o(\ln k)$ is negligible in asymptotics. This already shows that an increase in the number of coordinates in each of the vectors we deal with is quite promising.

*Proof of Theorem 9.* We take $3k$ instead of $k$ in Theorem 8 and put $s = 0$ and $a = b = 1/3$. Let $l = [\sqrt{k}/\ln k]$ and consider the following solutions to system (4.4): for $j \notin \{k-l, k-l+1, \dots, k+l-1, k+l\}$ we put $n'_j = 0$, otherwise, $n'_j c_j = 1/(2l+1)$, where $c_j = C_{3k}^j C_{3k-j}^{j+s} = C_{3k}^j C_{3k-j}^j$. Indeed,

$$\sum_j n'_j c_j = \sum_{j=k-l}^{k+l} \frac{1}{2l+1} = 1, \qquad \sum_j j n'_j c_j = \sum_{j=k-l}^{k+l} j \frac{1}{2l+1} = k = \frac{3k}{3} = 3kb.$$

It is easily seen that for our choice of the parameters $u = 3$. Let us find $v$. Note that for any $j_1, j_2 \in \{k-l, k-l+1, \dots, k+l-1, k+l\}$ we have $c_{j_1} \sim c_{j_2}$. This can be demonstrated by standard calculations with due account taken of the relation $l = o(\sqrt{k})$. Moreover, since $l$ has a particular form, one can find a function $\psi(k)$ which tends to zero with an increase in $k$ and satisfies the relation $|c_{j_1}/c_{j_2} - 1| = |n'_{j_2}/n'_{j_1} - 1| < \psi(k)$. Hence,

$$\prod_j (n'_j)^{n'_j c_j} = (1+o(1)) \prod_{j=k-l}^{k+l} (n'_k)^{1/(2l+1)} = (1+o(1)) n'_k \quad \Longrightarrow \quad v = c_k(2l+1).$$

As a result, (by Stirling's formula) we have $v \sim 3^{3k}(2l+1)/(ck)$ for some constant $c > 0$, whence it follows that $v \geqslant 3^{3k}/(c'\sqrt{k}\ln k)$ and

$$\log_u v \geqslant \log_3 \frac{3^{3k}\ln k}{c'\sqrt{k}} = 3k - \frac{1}{2}\log_3 k + O(\ln\ln k).$$

The proof of the theorem is complete.

Evidently, the choice of the parameters in Theorem 9 is not optimal. Therefore, to obtain the best possible estimates, we need to perform a rather sophisticated optimization (compare this with the remark that follows the statement of Proposition 6). Such optimization was carried out by I. M. Mitricheva using computer-based calculations: for small values of $k$ she obtained the following evaluations of the estimate established in Theorem 8 (see Table 3).

One can see that each time the maximum value (highlighted in bold) is attained at $s = 0$, and thus the estimate of Theorem 9 is not that far from optimality. It is also apparent that the new estimates are significantly sharper than those derived in § 3.3.4.

Table 3

| $s \setminus k$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| 0 | **1.8404** | **2.7207** | **3.6253** | **4.5461** | **5.4785** | **6.4195** | **7.3672** | **8.3203** |
| 1 | 1.8130 | 2.6986 | 3.6084 | 4.5329 | 5.4679 | 6.4108 | 7.3599 | 8.3140 |
| 2 | – | 2.6287 | 3.5545 | 4.4911 | 5.4347 | 6.3837 | 7.3373 | 8.2948 |
| 3 | – | – | 3.4489 | 4.4127 | 5.3738 | 6.3349 | 7.2972 | 8.2612 |
| 4 | – | – | – | 4.2740 | 5.2738 | 6.2578 | 7.2354 | 8.2102 |
| 5 | – | – | – | – | 5.1037 | 6.1379 | 7.1436 | 8.1367 |
| 6 | – | – | – | – | – | 5.9374 | 7.0047 | 8.0312 |
| 7 | – | – | – | – | – | – | 6.7747 | 7.8740 |
| 8 | – | – | – | – | – | – | – | 7.6153 |

4.3.5. *On chromatic numbers.* It is more or less clear that here one should expect to see much the same situation as in § 4.2.3. And it is indeed so: actually, we can deal only with the quantities $\widetilde{\kappa}^{\chi-\exp}_{\text{clique}}(k)$ and $\widetilde{\chi}^{\kappa}_{\text{clique}}(k)$. For the first of these quantities, a complete analogue of Theorem 6 holds.

**Theorem 10.** *The quantities $\widetilde{\kappa}^{\chi-\exp}_{\text{clique}}(k)$ obey exactly the same lower bounds as the quantities $\widehat{\kappa}_{\text{clique}}(k)$. In other words,*

$$\widetilde{\kappa}^{\chi-\exp}_{\text{clique}}(3) \geqslant 1.8404, \qquad \widetilde{\kappa}^{\chi-\exp}_{\text{clique}}(4) \geqslant 2.7207, \qquad \widetilde{\kappa}^{\chi-\exp}_{\text{clique}}(k) \geqslant 3.6253, \qquad \dots .$$

*Proof.* We know that the best possible estimates for the quantity $\widehat{\kappa}_{\text{clique}}(k)$ are obtained for $a = b$ (see § 4.3.4). Therefore, we can reproduce the proof of Theorem 6 almost verbatim. The only difference lies in the choice of the prime number $p$. In Theorem 6 it was taken as the least prime such that $2an - 4p < -an$. In that case $x = a$. However, here we have $x = 2a/(k-1)$ and thus we should take the least $p$ satisfying the condition $2an - 4p < -2a/(k-1)n$, so that $p \sim ak/(2(k-1)n)$. As a result, we have

$$\chi(H_n) \geqslant \frac{C_{2an}^{an}}{\sum_{i=0}^{p-1} C_{2an}^{i}} = (c(a,k) + o(1))^n,$$

where the value of $c(a,k)$ is larger than 1 for any admissible $a$ and $k$. The proof of the theorem is complete.

In estimating the quantity $\widetilde{\chi}^{\kappa}_{\text{clique}}(k)$ we can proceed in the same way as in Theorem 7: for a given value of $k$ optimize the quantity $c(a,k)$ introduced in the proof of Theorem 10 with respect to $a$. At the same time it should be taken into account that in our case (when $s = 0$) we have $2\lceil ak \rceil \leqslant k$, which means that $a \leqslant 1/k[k/2]$. Thus, we have established the following theorem.

**Theorem 11.** *The following estimate holds*:

$$\widetilde{\chi}^{\kappa}_{\text{clique}}(k) \geqslant \max_{a \leqslant 1/k[k/2]} c(a,k) =: c(k).$$

In particular, for $k = 3$ we arrive exactly at the bound 1.021 obtained in Theorem 7. One can further vary the value of $s$, but it presents severe technical difficulties, and we shall not perform such optimization. Instead, we shall prove one more theorem which, like Theorem 11, relies on the linear algebra method, but employs it in a somewhat different way. And in most situations the estimate derived in this new theorem will be sharper than the one in Theorem 11.

Thus, suppose that $a \leqslant 1/k[k/2]$ and let $p$ be the smallest prime such that $2an - p < -2a/(k-1)n$; so, $p \sim 2ak/(k-1)n$. Denote

$$S(a,k,n) = \frac{C_n^{2an} C_{2an}^{an}}{\sum_i C_n^i C_{n-i}^{p-1-2i}},$$

where summation in the denominator is taken over all admissible values of $i$. It can be shown by standard calculations that for any $k > 3$ we have

$$\sup_a S(a,k,n) = (c'(k) + o(1))^n,$$

where the function $c'(k)$ satisfies $c'(k) > 1$ and depends only on $k$. Here, we take supremum rather than maximum, since it is assumed, as usual, that $a \in \mathbb{Q}$ (otherwise the quantity $S(a,k,n)$ is not defined either).

**Theorem 12.** *Let $k > 3$ and $c'(k)$ be the function introduced above. Then*

$$\widetilde{\chi}^{\kappa}_{\text{clique}}(k) \geqslant c'(k).$$

*Proof.* Let us fix $k > 3$, $\varepsilon > 0$, and a quantity $a \in \mathbb{Q}$ for which

$$S(a,k,n) \geqslant (c'(k) - \varepsilon + o(1))^n.$$

We put $y = 2an - p \sim -2a/(k-1)n$. As in Theorem 6, consider the graphs $H_n \in \mathscr{A}'(n,2)$ determined by the parameters $a$, $b = a$, and by the forbidden inner product $y$, as well as the corresponding graphs $G_n \in \mathscr{A}(n,2,r')$. Since $y \sim -2a/(k-1)n$, the graphs $G_n$ have the desired asymptotic characteristics: they exhibit the appropriate relationship between the number of vertices and the number of $k$-cliques (that is, $\kappa > 1$) and $r' \sim r_{\text{clique}}(k)$. Thus, if we demonstrate that

$$\chi(G_n) = \chi(H_n) \geqslant S(a,k,n),$$

then, due to the arbitrary choice of $\varepsilon$, it will prove the theorem.

We proceed as in the proof of Theorem 6:

$$\chi(H_n) \geqslant \frac{|V(H_n)|}{\alpha(H_n)} = \frac{C_n^{2an} C_{2an}^{an}}{\alpha(H_n)},$$

with the only difference being that the denominator is estimated in another way by $\sum_i C_n^i C_{n-i}^{p-1-2i}$. To do this we take the polynomials $P_{\mathbf{x}} \in \mathbb{Z}_p[y_1,\dots,y_n]$ that correspond to the vertices $\mathbf{x}$ of the graph $H_n$, $\mathbf{x} = (x_1, \dots, x_n)$,

$$P_{\mathbf{x}}(\mathbf{y}) = \prod_{j\in J}(j - (\mathbf{x}, \mathbf{y})), \qquad J = \{1, \dots, p\} \setminus \{2an \mod p\},$$
$$\mathbf{y} = (y_1, \dots, y_n), \qquad y_l^3 = y_l, \qquad l \in \{1, \dots, n\}.$$

The degree of each polynomial of this kind is $p-1$, and owing to the equalities $y_l^3 = y_l$ the degree of such a polynomial in each of the variables is at most two. Thus, all the polynomials $P_{\mathbf{x}}$ lie in a space of dimension $\sum_i C_n^i C_{n-i}^{p-1-2i}$. We take an arbitrary independent set of vertices $W = \{\mathbf{x}_1, \dots, \mathbf{x}_s\}$ (that is, $(\mathbf{x}_i, \mathbf{x}_j) \neq 2an - p$ for $i \neq j$) and verify that the polynomials $P_{\mathbf{x}_1}, \dots, P_{\mathbf{x}_s}$ are linearly independent over $\mathbb{Z}_p$. Thus, the required estimate is established. The proof of Theorem 12 is complete.

First, let us compare the results provided by Theorems 11 and 12. To do this we present a table (Table 4) of approximate evaluations of $c(k)$ and $c'(k)$.

Table 4

| $k$ | $c(k)$ – Theorem 11 | $c'(k)$ – Theorem 12 |
|---|---|---|
| 3 | **1.0212** | – |
| 4 | **1.0583** | 1.0266 |
| 5 | **1.0594** | 1.0519 |
| 6 | **1.0858** | 1.0694 |
| 7 | 1.0797 | **1.0819** |
| 8 | **1.0995** | 1.0912 |
| 9 | 1.0920 | **1.0985** |
| 10 | **1.1077** | 1.1042 |
| 11 | 1.1002 | **1.1089** |
| 12 | **1.1131** | 1.1128 |
| 13 | 1.1059 | **1.1161** |
| 14 | 1.1170 | **1.1189** |
| 15 | 1.1102 | **1.1213** |

A certain odd behaviour of the quantities highlighted in bold (provided sometimes by $c(k)$ and sometimes by $c'(k)$) is due to the fact that $a \leqslant 1/k[k/2]$ and that this bound exhibits jumps as the parity of $k$ is altered. Nevertheless, it is easily verified that $c(k) \to 1/2 \cdot 3^{3/4} = 1.139\dots$ and $c'(k) \to 2/\sqrt{3} = 1.154\dots$. Therefore the apparent superiority of the estimate established in Theorem 12, which starts with $k = 13$, is kept for all $k \geqslant 13$.

In § 3.3.5, $\{0, 1\}$-vectors were also used to derive estimates for the quantities $\widetilde{\chi}^{\kappa}_{\text{clique}}(k)$. It is curious that for odd values of $k > 3$ they are better than the estimates established in Theorem 11, for $k = 3$ only Theorem 11 provides an estimate, and for even values of $k$ the estimates coincide. Thus, all three results are of their own importance: Theorem 11 for $k = 3$, Theorem 4 for small values of $k \neq 3$, and Theorem 12 for large values of $k$ (see Table 5).

Table 5

| $k$ | Theorem 4 | $c(k)$ – Theorem 11 | $c'(k)$ – Theorem 12 |
|---|---|---|---|
| 3 | – | **1.0212** | – |
| 4 | **1.0583** | **1.0583** | 1.0266 |
| 5 | **1.0641** | 1.0594 | 1.0519 |
| 6 | **1.0858** | **1.0858** | 1.0694 |
| 7 | **1.0883** | 1.0797 | 1.0819 |
| 8 | **1.0995** | **1.0995** | 1.0912 |
| 9 | **1.1008** | 1.0920 | 1.0985 |
| 10 | **1.1077** | **1.1077** | 1.1042 |
| 11 | 1.1085 | 1.1002 | **1.1089** |
| 12 | **1.1131** | **1.1131** | 1.1128 |
| 13 | 1.1137 | 1.1059 | **1.1161** |
| 14 | 1.1170 | 1.1170 | **1.1189** |
| 15 | 1.1174 | 1.1102 | **1.1213** |
| $\infty$ | $3^{3/4}/2 = 1.139\dots$ | $3^{3/4}/2 = 1.139\dots$ | $\mathbf{2/\sqrt{3} = 1.154\dots}$ |

**4.4. Results of the section.** Let us briefly summarize the results of this section following the lines of the summary in § 3.4.

1. We have proved Theorem 8 (see § 4.3.3), in which, as in Theorem 2 in § 3.3.3, we established a lower bound for the quantity $\widehat{\kappa}_{\text{clique}}(k)$ and which, just as Theorem 2, is optimal for its particular case (§§ 4.3.1, 4.3.2).
2. Owing to the results mentioned in item 1 we have improved all results in item 2 of § 3.4: in Table 3 in § 4.3.4 all the best estimates exceed their analogues in Table 1 in § 3.3.4, and Theorem 9 in § 4.3.4 establishes sharper asymptotic inequalities than Theorem 3 in § 3.3.4.
3. The problem mentioned in item 3 of § 3.4 is completely rectified. Namely, in Theorems 6 and 7 in § 4.2.3 we obtained nontrivial estimates for the quantities $\widetilde{\kappa}^{\chi-\exp}_{\text{clique}}(3)$ and $\widetilde{\chi}^{\kappa}_{\text{clique}}(3)$.
4. Two nontrivial analogues of Theorem 4 in § 3.3.5 have been established: these are Theorems 11 and 12 (§ 4.3.5), which improve Theorem 4 both for most specific values of $k$ (see Tables 4, 5) and as $k \to \infty$.
5. It has been demonstrated that, as in the $\{0, 1\}$-case, the quantities $\widetilde{\kappa}^{\chi-\exp}_{\text{clique}}(k)$ can hardly be distinguished from the quantities $\widehat{\kappa}_{\text{clique}}(k)$ (Theorem 10 in § 4.3.5).

## § 5. Constructions with $\{0, 1, \dots, m\}$-vectors

This section is organized in a somewhat different way from the previous two sections. Namely, we do not discuss triangles separately. In other respects it has the same structure. Subsection 5.1 is devoted to the general properties of graphs of the class $\mathscr{A}(n,m)$, in § 5.2 we discuss arbitrary cliques, and in § 5.3 we give a general summary.

**5.1. General properties of a graph of the class $\mathscr{A}(n,m)$.** We note straight away that the cases of the equalities $m = 1$ and $m = 2$ are allowed. Moreover, in the last case we shall speak formally of somewhat different objects than those discussed in the previous section. Therefore, it should be expected that in the case $m = 1$ the results will agree completely with those of § 3, and for $m = 2$ they may be reduced to the results in § 4 by simple transformations.

Now the line of reasoning is quite standard. If a graph $G$ belongs to the class $\mathscr{A}(n,m)$, then each of its vertices is determined by the parameters $l_0, l_1, \dots, l_m$, and we assume, as everywhere above, that $l_i = a_i n$, where $a_i \in (0,1) \cap \mathbb{Q}$ and $a_0 + \dots + a_m = 1$. The forbidden inner product will be denoted by $xn$, as it was in § 3, but not in § 4.

It is evident that the graphs determined by the parameters $a_i$ and $x$ lie on a sphere centred at the point

$$\biggl(\sum_{i=0}^m ia_i, \dots, \sum_{i=0}^m ia_i\biggr).$$

The square of the radius of such a sphere equals

$$\sum_{j=0}^m a_j n\biggl(j - \sum_{i=0}^m ia_i\biggr)^2 = \sum_{j=0}^m j^2 a_j n - \biggl(\sum_{j=0}^m ja_j\biggr)^2 n.$$

At the same time, the squared length of the edge of our graph equals $2\sum_{j=0}^m j^2 a_j n - 2xn$, which for the corresponding (normalized) graph $H \in \mathscr{A}(n,m,r')$ gives

$$r' = \biggl(\frac{\sum_{j=0}^m j^2 a_j - (\sum_{j=0}^m ja_j)^2}{2(\sum_{j=0}^m j^2 a_j - x)}\biggr)^{1/2}. \tag{5.1}$$

This formula is in good agreement with formula (3.2); it may be reduced to formula (4.1) by changing $j = 0, 1, 2$ to $j = -1, 0, 1$ and $x$ to $-x$.

**5.2. Cliques with an arbitrary $k$.** In the previous sections we have completely paved the way for formulating the most general results. Therefore, we take any $k \geqslant 3$ and proceed with it. The sequence of subsubsections in this subsection repeats that in § 4.3.

5.2.1. *Necessary and sufficient conditions for the existence of a $k$-clique.* The following assertion holds.

**Proposition 7.** *Suppose that $k \geqslant 3$ and let a graph $G_n \in \mathscr{A}(n, m, r_{\text{clique}}(k))$ be obtained by normalization of a graph $H_n \in \mathscr{A}(n, m)$ determined by the parameters $\mathbf{a} = (a_0, \dots, a_m)$ and $x$, and at the same time $\omega(G_n) = \omega(H_n) = k$, which is to say that the graphs $G_n$ and $H_n$ contain $k$-cliques. Then the quantity $\sum_{i=0}^m ia_i$ must have the form $\sum_{i=0}^m ia_i = s/k$ for some (arbitrary) $s \in \mathbb{N}$ and the corresponding value of $x$ must be equal to*

$$\frac{k(\sum_{i=0}^m ia_i)^2 - \sum_{i=0}^m i^2 a_i}{k-1}.$$

*Moreover, the system*

$$\begin{cases} \displaystyle\sum_{i=0}^m iq_i = s, \\ \displaystyle\sum_{i=0}^m q_i = k \end{cases} \tag{5.2}$$

*should have a solution in nonnegative integers $q_i$, $i \in \{0, \dots, m\}$, and the vector $k\mathbf{a}$ should be contained in the convex hull of the set of all its solutions $\mathbf{Q} = \{\mathbf{q}^j\} = \{(q_0^j, \dots, q_m^j)\}$.*

Let us find out why the similar Propositions 1 and 5 are particular cases of Proposition 7. First, consider the $\{0,1\}$-case. Then $a_0 = 1 - a$ and $a_1 = a$, which means that $\mathbf{a} = (1-a, a)$. In Proposition 1 we write $a = i/k$, $i \in \{1, \dots, k-1\}$ and in Proposition 7 it is claimed that $\sum_{i=0}^m ia_i = s/k$. However, now we have $\sum_{i=0}^m ia_i = a$, and so it is also required that $a = s/k$. Of course, here $s \in \{1, \dots, k-1\}$ too since we have $a \in (0,1)$. Further, in Proposition 1 we have $x = i(i-1)/(k(k-1))$. However, in Proposition 7 we also have

$$x = \frac{ka^2 - a}{k-1} = \frac{s^2/k - s/k}{k-1} = \frac{s(s-1)}{k(k-1)},$$

and so they completely agree. Thus, Proposition 1 is a corollary of Proposition 7. We note that in the $\{0,1\}$-case the condition $\sum_{i=0}^m ia_i = s/k$ is equivalent to the condition that the vector $k\mathbf{a} = k(1-a, a)$ belongs to the convex hull of the set $\mathbf{Q}$ since the only solution to system (5.2) is equal to the vector $\mathbf{q}^1 = (k-s, s)$.

Let us study the relationship between Proposition 5 and Proposition 7. Proposition 5 does not deal with the $\{0,1,2\}$-case, but rather with the $\{-1,0,1\}$-case. Therefore in Proposition 7 we have to make a formal change $i \mapsto i-1$. The condition $\sum_{i=0}^m ia_i = s/k$ reduces to the condition $-a_{-1} + a_1 = s/k$, and this agrees well with the condition $a - b = s/k$. Further, $x$ in Proposition 5 corresponds to $-x$ in Proposition 7. With the notation of Proposition 5 the corresponding assertion of Proposition 7 has the form

$$-x = \frac{k(\sum_{i=0}^m ia_i)^2 - \sum_{i=0}^m i^2 a_i}{k-1} = \frac{k(a-b)^2 - b - a}{k-1}$$
$$\iff \quad x = \frac{a + b - k(a-b)^2}{k-1},$$

and we again arrive at a complete concordance. It remains to show that we necessarily have $s \in \{0, \dots, k-2\}$ and $2\lceil bk \rceil \leqslant k - s$. It is clear that $s < k$, since for

$s \geqslant k$ we have $a - b \geqslant 1$, which cannot happen. So why do we have $s \neq k-1$ and $2\lceil bk \rceil \leqslant k - s$? Here, the condition involving system (5.2) becomes important. In the $\{-1, 0, 1\}$-notation it appears as follows:

$$\begin{cases} -q_{-1} + q_1 = s, \\ q_{-1} + q_0 + q_1 = k. \end{cases} \tag{5.3}$$

First, let us assume that $s = k - 1$. Then the only solution to (5.3) is equal to the vector $\mathbf{q}^1 = (0, 1, k-1)$. At the same time

$$k\mathbf{a} = k(b, 1 - a - b, a) = (bk, -2bk + 1, bk + k - 1).$$

Such a vector $k\mathbf{a}$ can be expressed in terms of $\mathbf{q}^1$ only if $k = 0$, which is impossible. Hence, $s \neq k-1$. Second, let us subtract the first equation in (5.3) from the second: $2q_{-1} + q_0 = k - s$. Consequently, $2q_{-1} \leqslant k - s$. We know that $k\mathbf{a} = \sum_j v^j \mathbf{q}^j$ and, moreover, $\sum_j v^j \leqslant 1$ and $v^j \geqslant 0$. Write this condition for the first coordinate:

$$bk = \sum_j v^j q_{-1}^j \leqslant \max_j q_{-1}^j \sum_j v^j \leqslant \max_j q_{-1}^j \quad \Longrightarrow \quad 2\lceil bk \rceil \leqslant 2 \max_j q_{-1}^j \leqslant k - s.$$

As a result, we see that the conditions of Proposition 5 follow from the conditions of Proposition 7. The inverse implication is verified similarly and thus Proposition 5 is a particular case of Proposition 7.

*Proof of Proposition* 7. We follow the lines of the proofs of Propositions 1 and 5. Consider an arbitrary $k$-clique $K_k \subset H_n$. Its vectors form a matrix $\mathscr{M}_k$: this matrix contains $k$ rows and $n$ columns and its entries are integers $i \in \{0, 1, \dots, m\}$. Denote by $q_i^j$ the number of entries equal to $i$ in the column with index $j \in \{1, \dots, n\}$ and let $s^j$ be the total sum of all entries in the $j$th column. Certainly, $s^j = \sum_{i=0}^m i q_i^j$. As usual, we calculate the total sum of all entries of the matrix $\mathscr{M}_k$ in two ways:

$$\sum_{j=1}^n s^j = k \sum_{i=0}^m i a_i n.$$

Now consider the sum of the pairwise inner products of the matrix row vectors. On the one hand, it certainly equals $C_k^2 xn$, since $K_k$ is a clique. On the other hand, denote by $d^j$ the contribution to this sum made by the $j$th column. Then

$$\sum_{j=1}^n d^j = C_k^2 xn.$$

Let us find $d^j$, $j = 1, \dots, n$. To do this we extend the $j$th column by replacing each entry equal to $i \in \{1, \dots, m\}$ by $i$ successive entries equal to 1. Now if we sum up the pairwise products of different entries of the new column, then we obtain a number which exceeds the desired contribution by the value $\sum_{i=0}^m C_i^2 q_i^j$. At the same time, this number equals $C_{s^j}^2$, whence it follows that

$$d^j = C_{s^j}^2 - \sum_{i=0}^m C_i^2 q_i^j.$$

Note also that $\sum_{j=1}^n q_i^j = ka_i n$. Thus, we have

$$\begin{cases} \sum_{j=1}^n s^j = k\sum_{i=0}^m ia_i n, \\ \sum_{j=1}^n d^j = C_k^2 xn \end{cases} \iff \begin{cases} \sum_{j=1}^n s^j = k\sum_{i=0}^m ia_i n, \\ \sum_{j=1}^n \Bigl(C_{s^j}^2 - \sum_{i=0}^m C_i^2 q_i^j\Bigr) = C_k^2 xn \end{cases}$$

$$\iff \begin{cases} \sum_{j=1}^n s^j = k\sum_{i=0}^m ia_i n, \\ \sum_{j=1}^n s^j(s^j-1) - \sum_{i=0}^m ki(i-1)a_i n = k(k-1)xn \end{cases}$$

$$\iff \begin{cases} \sum_{j=1}^n s^j = k\sum_{i=0}^m ia_i n, \\ \sum_{j=1}^n (s^j)^2 - \sum_{i=0}^m ki^2 a_i n = k(k-1)xn. \end{cases}$$

Now, as usual, we recall that the graph $G_n$ lies on a sphere of minimum radius. It follows from relation (5.1) that for given $a_i$, $i \in \{0,\dots,m\}$, we have to choose $x$ as small as possible. However, in view of the last system this means that for a given sum $\sum_{j=1}^n s^j$ we need to minimize the sum $\sum_{j=1}^n (s^j)^2$. Hence, we necessarily have $s^1 = s^2 = \dots = s^n = s$, and the first equation of the system gives $sn = kn\sum_{i=0}^m ia_i$ or $\sum_{i=0}^m ia_i = s/k$. Then the second equation of the system yields

$$x = \frac{(k\sum_{i=0}^m ia_i)^2 - k\sum_{i=0}^m i^2 a_i}{k(k-1)} = \frac{k(\sum_{i=0}^m ia_i)^2 - \sum_{i=0}^m i^2 a_i}{k-1}.$$

Moreover, substituting this expression for $x$ into (5.1) we obtain

$$\frac{\sum_{i=0}^m i^2 a_i - (\sum_{i=0}^m ia_i)^2}{2(\sum_{i=0}^m i^2 a_i - (k(\sum_{i=0}^m ia_i)^2 - \sum_{i=0}^m i^2 a_i)/(k-1))} = \frac{k-1}{2k} = r^2_{\text{clique}}(k).$$

It remains to demonstrate that the conditions involving system (5.2) should necessarily be satisfied. Recall that $s^j = \sum_{i=0}^m iq_i^j$. But it is known already that for some $s \in \mathbb{N}$ we have $s^j = s$, which means that for each $j$ the numbers $q_i^j$ obey the condition $\sum_{i=0}^m iq_i^j = s$. Moreover, it is evident that $\sum_{i=0}^m q_i^j = k$ and, therefore, if system (5.2) has no nonnegative integer solutions, then there is no way to obtain the numbers $q_i^j$ either. In other words, any vector $\mathbf{q} \in \mathbf{Q}$ is a potential set of the parameters $q_i^j$ characterizing the number of entries equal to $i$ in the $j$th column of the matrix $\mathscr{M}_k$. To avoid confusion we reassign indices $l \in \{1,\dots,n\}$ to the columns of $\mathscr{M}^k$. Let $\mathbf{q}^{j_l} \in \mathbf{Q}$ be the solution to system (5.2) whose components represent the aforementioned parameters for the $l$th column. Again, we note that $\sum_{l=1}^n q_i^{j_l} = ka_i n$, or, in the vector form, $\sum_{l=1}^n \mathbf{q}^{j_l} = k\mathbf{a}n$. Suppose that $\mathbf{q}^1 \in \mathbf{Q}$ occurs exactly $u_1$ times among the vectors $\mathbf{q}^{j_l}$, $\mathbf{q}^2 \in \mathbf{Q}$ occurs exactly $u_2$ times, and so on. We obtain a new relation

$$\sum_j u_j \mathbf{q}^j = k\mathbf{a}n \iff \sum_j \frac{u_j}{n}\mathbf{q}^j = k\mathbf{a}.$$

The last expression suggests that the vector $k\mathbf{a}$ lies in the convex hull of the vectors belonging to $\mathbf{Q}$: it is clear that $\sum u_j = n$ or $\sum u_j/n = 1$.

The proof of Proposition 7 is complete.

5.2.2. *Sufficient conditions for the existence of a $k$-clique.* The following assertion is valid.

**Proposition 8.** *Let $k \geqslant 3$ and $s \in \mathbb{N}$ and suppose that for the given values of the parameters $s$ and $k$ system (5.2) is solvable and the set of solutions of this system is $\mathbf{Q} = \{\mathbf{q}^j\} = \{(q_0^j, \dots, q_m^j)\}$, $j \in \mathbf{J}$. Further, suppose that numbers $a_0, \dots, a_m$ are chosen so as to satisfy the condition $\sum_{i=0}^m ia_i = s/k$ and let*

$$x = \frac{k(\sum_{i=0}^m ia_i)^2 - \sum_{i=0}^m i^2 a_i}{k-1}.$$

*Suppose that a graph $G_n \in \mathscr{A}(n, m, r_{\text{clique}}(k))$ is obtained by normalization of a graph $H_n \in \mathscr{A}(n, m)$ determined by the parameters $a_i$ and $x$. For each $j \in \mathbf{J}$ set*

$$r_j = \frac{k!}{\prod_{i=0}^m q_i^j!}$$

*and consider arbitrary nonnegative rational numbers $n'_j$ satisfying the system*

$$\begin{cases} \sum_{j\in\mathbf{J}} q_0^j n'_j r_j = ka_0, \\ \dots \\ \sum_{j\in\mathbf{J}} q_m^j n'_j r_j = ka_m. \end{cases} \tag{5.4}$$

*Suppose that $n$ is such that the numbers $n_j = n'_j n$ are integers. Then $\omega(G_n) = \omega(H_n) = k$.*

System (5.4) can be rewritten in vector form: $\sum_{j\in\mathbf{J}} \mathbf{q}^j n'_j r_j = k\mathbf{a}$. At the same time, combining all equations in (5.4) and taking into account the second equation in system (5.2) we obtain $\sum_{j\in\mathbf{J}} n'_j r_j = 1$. Thus, system (5.4) suggests that the vector $k\mathbf{a}$ lies in the convex hull of the vectors in the set $\mathbf{Q}$. And this means that Proposition 8 establishes the unimprovability of Proposition 7.

It should also be noted that system (5.4) is a direct generalization of system (4.4) in Proposition 6, which has a similar meaning. System (4.4) contains one less equation, but the point is that we can get rid of one of the equations in (5.4) due to the condition $\sum_{i=0}^m ia_i = s/k$. It is just that we have represented system (5.4) in a more symmetric form, which is more convenient for the comparison of Propositions 7 and 8. As a result, all remarks about Proposition 6 also apply to Proposition 8.

*Proof of Proposition* 8. As usual, we construct a matrix $\mathscr{M}_k$. To do this we choose nonzero roots of system (5.4) $n'_{j_1}, \dots, n'_{j_\mu}$ and consider the corresponding $n_{j_\nu}$. The matrix is composed of successive blocks. The first block is formed by all possible columns containing $q_0^{j_1}$ zero entries, $q_1^{j_1}$ entries equal to 1, …, $q_m^{j_1}$ entries equal to $m$. There are $r_{j_1}$ columns. The block described above is taken $n_{j_1}$ times. Then we similarly compose another block, corresponding to the index $j_2$, and take it $n_{j_2}$ times. And we proceed in this way up to $j_\mu$. It is evident that the matrix $\mathscr{M}_k$ is well defined. The composition of this matrix is identical with that of the matrix denoted in the same way in the proof of Proposition 6. And a verification of the fact that this matrix defines the desired clique is carried out in exactly the same way. So, we need hardly reproduce it once again here. The proof of Proposition 8 is complete.

5.2.3. *Estimating the number of $k$-cliques.* The following assertion holds.

**Theorem 13.** *Let $k \geqslant 3$ and $s \in \mathbb{N}$ and suppose that for the given values of the parameters $s$ and $k$ system (5.2) is solvable and the set of solutions to this system is $\mathbf{Q} = \{\mathbf{q}^j\} = \{(q_0^j, \dots, q_m^j)\}$, $j \in \mathbf{J}$. Further, suppose that numbers $a_0, \dots, a_m$ are chosen so as to satisfy the condition $\sum_{i=0}^m ia_i = s/k$. For each $j \in \mathbf{J}$ we put*

$$r_j = \frac{k!}{\prod_{i=0}^m q_i^j!}$$

*and consider arbitrary nonnegative rational numbers $n'_j$ satisfying (5.4). Denote*

$$u = \frac{1}{\prod_{i=0}^m a_i^{a_i}}, \qquad v = \frac{1}{\prod_{j\in\mathbf{J}} (n'_j)^{n'_j r_j}}.$$

*Here $(n'_j)^{n'_j} = 1$ as long as $n'_j = 0$. Then*

$$\widehat{\kappa}_{\text{clique}}(k) \geqslant \widehat{\kappa}_{\text{clique}}(k, m) \geqslant \log_u v.$$

It is easily seen that Theorem 13 is established in the same way as Theorem 8, and thus we omit its proof here.

5.2.4. *Optimization in Theorem 13.* The following assertion is valid.

**Theorem 14.** *The following inequality holds*:

$$\widehat{\kappa}_{\text{clique}}(k) \geqslant k + O\left(\frac{\ln k}{\ln \ln k}\right).$$

In Theorem 14 we significantly strengthen the results of Theorems 3 and 9: while earlier we reduced $k$ by the logarithm of $k$, now we subtract something that is the iterated logarithm times less. This result is obtained due to the optimization performed in Theorem 13; here, it is essential even to choose the value of the parameter $m$ appropriately.

*Proof of Theorem 14.* It may be assumed that $k$ is sufficiently large. We put $m = [\ln k/\ln \ln k]$, $k_1 = [k/(m+1)]$, $k_2 = k_1(m+1)$, and note that $k_2 \leqslant k$ and

$$k_2 \geqslant \left(\frac{k}{m+1} - 1\right)(m+1) = k - m - 1 = k - \frac{\ln k}{\ln \ln k}(1 + o(1)). \tag{5.5}$$

Let us apply Theorem 13 with $k_2$ substituted for the parameter $k$ involved therein. In the conditions of Theorem 13 we put

$$s = \frac{k_2 m}{2} = \frac{k_1 m(m+1)}{2},$$

so that, obviously, $s \in \mathbb{N}$. Take $a_0 = a_1 = \cdots = a_m = 1/(m+1)$. Then in any case we have $\sum_{i=0}^m a_i = 1$ and

$$\sum_{i=0}^m i a_i = \frac{1}{m+1}\,\frac{m(m+1)}{2} = \frac{m}{2} = \frac{s}{k_2},$$

which was to be shown.

Now we need to take a close look at systems (5.2) and (5.4). Certainly, one of the solutions to system (5.2) is equal to the vector $\mathbf{q}^{j_0} = (k_1, \dots, k_1)$. Let us present a sufficiently large number of other solutions located 'symmetrically' about $\mathbf{q}^{j_0}$. Namely, we put $f = [\sqrt{k_1}/m^3]$ and let $\delta_2, \dots, \delta_m$ be arbitrary integers in the interval $[-f, f]$. Finally, we define

$$\delta_1 = -\sum_{i=2}^m i\delta_i, \qquad \delta_0 = -\sum_{i=1}^m \delta_i.$$

Then it is apparent that $\sum_{i=0}^m i\delta_i = 0$ and $\sum_{i=0}^m \delta_i = 0$, whence it follows that for any vector $\mathbf{q}^j = (k_1 + \delta_0, \dots, k_1 + \delta_m)$ the system of conditions (5.2) is fulfilled. Denote by $\mathbf{Q}' \subset \mathbf{Q}$ the set of all solutions to (5.2) thus obtained and let $\mathbf{Q}' = \{\mathbf{q}^j\}_{j\in\mathbf{J}'}$, where $\mathbf{J}' \subset \mathbf{J}$ and $j_0 \in \mathbf{J}'$. Putting $F = |\mathbf{J}'| = |\mathbf{Q}'|$ we have $F \geqslant (2f)^{m-1}$.

Let us turn to system (5.4). For each $j \in \mathbf{J}\backslash\mathbf{J}'$ we set $n'_j r_j = 0$; for other values of $j$ we set $n'_j r_j = 1/F$. We have to show that $\sum_{j\in\mathbf{J}'} q_i^j n'_j r_j = a_i k_2 = k_1$ for all values of $i$. Evidently, for each $m$-tuple $(\delta_0, \dots, \delta_m) \neq \mathbf{0}$ that specifies a vector $\mathbf{q}^j \in \mathbf{Q}'$, the $m$-tuple with the entries of opposite sign also specifies a vector $\mathbf{q}^l \in \mathbf{Q}'$. Cancelling such pairs in the sum $\sum_{j\in\mathbf{J}'} q_i^j n'_j r_j = 1/F \sum_{j\in\mathbf{J}'} q_i^j$ and taking into account that $|\mathbf{J}'| = F$ we obtain

$$\frac{1}{F}\sum_{j\in\mathbf{J}'} q_i^j = \frac{1}{F}\sum_{j\in\mathbf{J}'} k_1 = k_1,$$

as was to be shown.

All the parameters have been chosen. Let us find $u$ and $v$ and verify the inequality $\log_u v \geqslant k + O(\ln k/\ln\ln k)$. Clearly, we have $u = m+1$. Taking into account the constraint $f = o(\sqrt{k_1}/m^2)$ and the fact that $|q_i^j - k_1| \leqslant 2fm^2 = o(\sqrt{k_1})$ for all $i$ and $j \in \mathbf{J}'$, we obtain

$$r_j \sim r_{j_0} = \frac{(k_1(m+1))!}{(k_1!)^{m+1}}, \qquad n_j \sim n_{j_0}, \qquad v \sim F r_{j_0}.$$

Thus, for sufficiently large values of $k$, with due regard for Stirling's formula and relation (5.5), we have

$$\begin{aligned}
\log_u v &= \log_{m+1}((1+o(1)) r_{j_0} F) \geqslant O(1) + \log_{m+1}\biggl(\frac{(k_1(m+1))!}{(k_1!)^{m+1}}(2f)^{m-1}\biggr) \\
&\geqslant O(1) + \log_{m+1}\biggl(\frac{\sqrt{k_1}(k_1(m+1))^{k_1(m+1)}}{(10\sqrt{k_1})^{m+1} k_1^{k_1(m+1)}}(2f)^{m-1}\biggr) \\
&= O(1) + \log_{m+1}\biggl(\frac{(m+1)^{k_1(m+1)}}{10^{m+1} k_1^{m/2}}(2f)^{m-1}\biggr)
\end{aligned}$$

$$
\begin{aligned}
&= O(1) + k_1(m+1) - (m+1)\log_{m+1} 10 - \frac{m}{2}\log_{m+1} k_1 + (m-1)\log_{m+1} 2f \\
&= O(1) + k_2 - o(m) - \frac{m}{2}\log_{m+1} k_1 + (m-1)\log_{m+1} 2 + (m-1)\log_{m+1}\left[\frac{\sqrt{k_1}}{m^3}\right] \\
&= O(1) + k_2 - o(m) - \frac{m}{2}\log_{m+1} k_1 + \frac{m-1}{2}\log_{m+1} k_1 - (m-1)\log_{m+1} m^3 \\
&= k_2 - \frac{1}{2}\log_{m+1} k_1 + O(m) \geqslant k - \frac{\ln k}{\ln\ln k}(1+o(1)) + O\left(\frac{\ln k}{\ln\ln k}\right) \\
&= k + O\left(\frac{\ln k}{\ln\ln k}\right).
\end{aligned}
$$

Hence,

$$
\widehat{\kappa}_{\text{clique}}(k) \geqslant \widehat{\kappa}_{\text{clique}}(k_2) \geqslant \log_u v \geqslant k + O\left(\frac{\ln k}{\ln\ln k}\right).
$$

The proof of the theorem is complete.

It is seen from the next-to-last relation in the proof that the parameter $m$ is chosen optimally with respect to the order of growth of the remainder. If we take it smaller, then $O(m)$ will decrease, but $\log_{m+1} k_1$ will increase and vice versa. In this connection it seems interesting to consider upper estimates for the quantity $\widehat{\kappa}_{\text{clique}}(k)$. At the same time, it seems unlikely that our choice of the parameters is globally optimal. Such an optimization problem is very hard to solve even asymptotically. And a straightforward exhaustive search is hardly feasible here. It is quite reasonable to expect that tables like the one in § 4.3.4 will contain better and better results. However, compiling such tables is very laborious and we shall not do it here.

5.2.5. *On chromatic numbers.* In this subsubsection we actually say nothing new as compared to § 4.3.5. First, we have a complete analogue of Theorem 10, which states that the estimates for the quantities $\widehat{\kappa}_{\text{clique}}(k)$ obtained above are carried over as well to the quantities $\widehat{\kappa}^{\chi-\exp}_{\text{clique}}(k)$.

Second, one can formulate and prove a great number of theorems similar to Theorems 11 and 12. The problem is that each of these theorems involves a lot of parameters and we have not succeeded in choosing the values of these parameters so as to improve the results shown in the tables presented in § 4.3.5. Nevertheless, we give below a formulation which in a sense is parallel to the formulation of Theorem 12 and which appears most promising from the viewpoint of final optimization.

Thus, suppose that numbers $k$, $m$, $a_0, \dots, a_m$, $x$ satisfy all the conditions of Proposition 8. Take the least prime $p$ such that $(a_1 + 4a_2 + 9a_3 + \dots + m^2 a_m)n - p < xn$ and denote

$$
S(m, \mathbf{a}, k, n) = \frac{n!/((a_0 n)! \cdots (a_m n)!)}{\sum_{\mathbf{u}\in\mathscr{U}} n!/(u_0! \cdots u_m!)},
$$

where

$$
\mathscr{U} = \bigl\{\mathbf{u} = (u_0, \dots, u_m) \in \mathbb{N}^{m+1} : u_0 + \dots + u_m = n,\ u_1 + 2u_2 + \dots + m u_m \leqslant p-1\bigr\}.
$$

Let $c(k,m)$ be defined by the relation

$$\max_{\mathbf{a}} S(m,\mathbf{a},k,n) = (c(k,m)+o(1))^n.$$

Here, we do not rule out the possibility of $c(k,m)=1$. The following assertion holds.

**Theorem 15.** *The following estimate is valid*:

$$\widetilde{\chi}^{\kappa}_{\text{clique}}(k) \geqslant \widetilde{\chi}^{\kappa}_{\text{clique}}(k,m) \geqslant c(k,m).$$

*Proof*. The proof is quite standard (cf. the proof of Theorem 12). The forbidden inner product $y$ is taken to be $(a_1+4a_2+9a_3+\dots+m^2a_m)n-p$. The numerator in the definition of the quantity $S(m,\mathbf{a},k,n)$ is $|V(H_n)|$, the denominator is equal to the estimate for the independence number obtained with the use of the polynomial technique. Here the polynomials $P_{\mathbf{x}} \in \mathbb{Z}_p[y_1,\dots,y_n]$ that correspond to the vertices $\mathbf{x}=(x_1,\dots,x_n)$ of graph $H_n$ are as follows:

$$P_{\mathbf{x}}(\mathbf{y}) = \prod_{j\in J}(j-(\mathbf{x},\mathbf{y})), \qquad J=\{1,\dots,p\}\setminus\{y \mod p\}, \quad \mathbf{y}=(y_1,\dots,y_n),$$
$$y_l(y_l-1)(y_l-2)\cdots(y_l-m)=0, \qquad l\in\{1,\dots,n\}.$$

The degree of any such polynomial is at most $p-1$, and its degree in each of the variables is at most $m$. Thus, $u_i$ is the number of variables occurring to power $i$ in some monomial. The proof of the theorem is complete.

**5.3. Results of the section.** In this case the results are quite apparent.

1. We have proved Theorem 13, which is optimal for each value of $m$.
2. We have proved Theorem 14, which provides a new estimate for the remainder in the asymptotic representation of the quantity $\widehat{\kappa}_{\text{clique}}(k)$.
3. The validity of Theorem 15 has been established, which will probably improve the results of Theorems 11 and 12 on chromatic numbers.
4. It is still interesting to estimate the quantity $\widehat{\kappa}_{\text{clique}}(k)$ from above.
5. Optimization in Theorem 15 is the subject of further study.

## Bibliography


[1] P. Brass, W. Moser and J. Pach, *Research problems in discrete geometry*, Springer-Verlag, New York 2005.

[2] A. Soifer, *The mathematical coloring book*, Springer-Verlag, New York 2009.

[3] A. Soifer, "Chromatic number of the plane: its past, present and future", *Mat. Prosveshchenie*, 2004, no. 8, 186–221. (Russian)

[4] H. Hadwiger, "Ein Überdeckungssatz für den Euklidischen Raum", *Portugaliae Math.* **4** (1944), 140–144.

[5] F. Harary, *Graph theory*, Addison-Wesly, Reading, MA 1969.

[6] A. M. Raigorodskii, "Borsuk's problem and the chromatic numbers of some metric spaces", *Uspekhi Mat. Nauk* **56**:1 (2001), 107–146; English transl. in *Russian Math. Surveys* **56**:1 (2001), 103–139.

[7] A. M. Raigorodskii, *Chromatic numbers*, Moscow Center for Continuous Mathematical Education, Moscow 2003. (Russian)

[8] A. M. Raigorodskii, *Linear algebraic method in combinatorics*, Moscow Center for Continuous Mathematical Education, Moscow 2007. (Russian)

[9] A. M. Raigorodskii, "Coloring distance graphs and graphs of diameters", *Thirty essays on geometric graph theory*, Lecture Notes in Math., Springer-Verlag, New York 2013, pp. 429–460.

[10] P. K. Agarwal and J. Pach, *Combinatorial geometry*, Wiley-Intersci. Ser. Discrete Math. Optim., John Wiley and Sons Inc., New York 1995.

[11] V. Klee and S. Wagon, *Old and new unsolved problems in plane geometry and number theory*, Dolciani Math. Exp., vol. 11, Math. Association of America, Washington 1991.

[12] L. A. Székely, "Erdős on unit distances and the Szemerédi–Trotter theorems", *Paul Erdős and his mathematics*, vol. II (Budapest, Hungary 1999), Bolyai Soc. Math. Stud., vol. 11, Springer-Verlag, Berlin 2002, pp. 649–666.

[13] A. M. Raigorodskii, "On the chromatic number of a space", *Uspekhi Mat. Nauk* **55**:2 (2000), 147–148; English transl. in *Russian Math. Surveys* **55**:2 (2000), 351–352.

[14] D. G. Larman and C. A. Rogers, "The realization of distances within sets in Euclidean space", *Mathematika* **19** (1972), 1–24.

[15] P. Frankl and R. M. Wilson, "Intersection theorems with geometric consequences", *Combinatorica* **1**:4 (1981), 357–368.

[16] A. M. Raigorodskii, "The Erdös–Hadwiger problem and the chromatic numbers of finite geometric graphs", *Mat. Sb.* **196**:1 (2005), 123–156; English transl. in *Sb. Math.* **196**:1 (2005), 115–146.

[17] A. M. Raigorodskii, "The problems of Borsuk and Grünbaum on lattice polytopes", *Izv. Ross. Akad. Nauk Ser. Mat.* **69**:3 (2005), 81–108; English transl. in *Izv. Math.* **69**:3 (2005), 513–537.

[18] E. S. Gorskaya, I. M. Mitricheva, V. Yu. Protasov and A. M. Raigorodskii, "Estimating the chromatic numbers of Euclidean space by convex minimization methods", *Mat. Sb.* **200**:6 (2009), 3–22; English transl. in *Sb. Math.* **200**:6 (2009), 783–801.

[19] P. Erdős and R. L. Graham, *Problem proposed at the* 6*th Hungarian combinatorial conference*, Eger July 1981.

[20] L. Lovaśz, "Self-dual polytopes and the chromatic number of distance graphs on the sphere", *Acta Sci. Math. (Szeged)* **45**:1–4 (1983), 317–323.

[21] J. Matoušek, *Using the Borsuk-Ulam theorem*, Universitext, Springer-Verlag, Berlin 2003.

[22] A. M. Raigorodskii, "On the chromatic numbers of spheres in Euclidean spaces", *Dokl. Ross. Akad. Nauk* **432**:2 (2010), 174–177; English transl. in *Dokl. Math.* **81**:3 (2010), 379–382.

[23] A. M. Raigorodskii, "On the chromatic numbers of spheres in $\mathbb{R}^n$", *Combinatorica* **32**:1 (2012), 111–123.

[24] E. E. Demekhin, A. M. Raigorodskii and O. I. Rubanov, "Distance graphs having large chromatic numbers and containing no cliques or cycles of a given size", *Mat. Sb.* **204**:4 (2013), 49–78; English transl. in *Sb. Math.* **204**:4 (2013), 508–538.

[25] A. M. Raigorodskii, "On distance graphs with large chromatic number but without large simplices", *Uspekhi Mat. Nauk* **62**:6 (2007), 187–188; English transl. in *Russian Math. Surveys* **62**:6 (2007), 1224–1225.

[26] A. M. Raigorodskii and O. I. Rubanov, "On the clique and the chromatic numbers of high-dimensional distance graphs", *Number theory and applications* (Allahabad, India 2007), Hindustan Book Agency, New Delhi 2009, pp. 149–157.

[27] A. M. Raigorodskii and O. I. Rubanov, "Distance graphs with large chromatic number and without large cliques", *Mat. Zametki* **87**:3 (2010), 417–428; English transl. in *Math. Notes* **87**:3 (2010), 392–402.

[28] P. Frankl and V. Rödl, "Forbidden intersections", *Trans. Amer. Math. Soc.* **300**:1 (1987), 259–286.

[29] R. C. Baker, G. Harman and J. Pintz, "The difference between consecutive primes, II", *Proc. London Math. Soc.* (3) **83**:3 (2001), 532–562.

**Andrey B. Kupavskii**
Faculty of Innovation and Higher Technology,
Moscow Institute of Physics and Technology;
Faculty of Mechanics and Mathematics,
Moscow State University
*E-mail*: kupavskii@ya.ru

Received 15/MAR/12 and 12/FEB/13
Translated by A. PANKRAT'EV

**Andrey M. Raigorodskii**
Faculty of Innovation and Higher Technology,
Moscow Institute of Physics and Technology;
Faculty of Mechanics and Mathematics,
Moscow State University
*E-mail*: mraigor@yandex.ru