Two notions of unit distance graphs

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Two definitions

There are two well-known definitions of distance graphs. The first one is the following:

Complete distance graphs

A finite graph $G = (V, E)$ is a *complete (unit) distance graph in* $\mathbb{R}^d$ if $V \subset \mathbb{R}^d$ and $E = \{(x, y), x, y \in \mathbb{R}^d, |x - y| = 1\}$.

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The second one is slightly different:

Distance graphs

A finite graph $G = (V, E)$ is a *(unit) distance graph in* $\mathbb{R}^d$ if it is a subgraph of some complete distance graph in $\mathbb{R}^d$.

Motivation. Erdős on unit distances

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In 1965 Erdős, Harary and Tutte introduced the concept of the Euclidean dimension:

Euclidean dimension $\dim G$ of a graph G is the minimum dimension d so that the graph G is isomorphic to some distance graph in $\mathbb{R}^d$.

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We can define analogous quantity in $\mathbb{R}^d$.
Formally,

$$\chi(\mathbb{R}^d) = \min\{m \in \mathbb{N} : \mathbb{R}^d = H_1 \cup \ldots \cup H_m : \\ \forall i, \forall x, y \in H_i \quad |x - y| \neq 1\}.$$

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Theorem(1951, P. Erdős, N.G. de Bruijn). If we accept the axiom of choice then the chromatic number of $\mathbb{R}^d$ is equal to the chromatic number of some *finite* distance graph in $\mathbb{R}^d$.

Distance and complete distance graphs. Notation

- $\mathcal{D}(d)$ — the set of all labeled distance graphs in $\mathbb{R}^d$
  $\mathcal{D}_n(d)$ — the set of all those of order n.
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Redefinition of quantities discussed above

$$f_2(n) = \max_{G\in\mathcal{D}_n(2)} |E(G)| = \max_{G\in\mathcal{CD}_n(2)} |E(G)|.$$

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For both quantities it makes no difference whether to consider distance or complete distance graphs.

Main theorem

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Theorem 1 (N. Alon, AK, 2013; AK, A. Raigorodskii, M. Titova, 2012)

1. For any $d \in \mathbb{N}$, we have $\log_2 |\mathcal{CD}_n(d)| \sim dn \log_2 n$.
2. For any $d \in \mathbb{N}, d \geqslant 4$, we have $\log_2 |\mathcal{D}_n(d)| \sim \left(1 - \frac{1}{[d/2]}\right) \frac{n^2}{2}$.

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Zero pattern of the P_j's at $x \in \mathbb{R}^l$ is $(\varepsilon_1, \ldots, \varepsilon_m) \in \{0,1\}^m$, where $\varepsilon_j = 0$, if $P_j(x) = 0$ and $\varepsilon_j = 1$ if $P_j(x) \neq 0$.

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L. Ronyai, L. Babai, M.K. Ganapathy, 2001, Theorem 1.3 and Corollary 1.5

Let $P_1, \ldots, P_m$ be m real polynomials in l real variables, and suppose the degree of each P_j does not exceed k. Then $z(P_1, \ldots, P_m) \leqslant \binom{km}{l}$.

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Each labeled complete distance graph in $\mathbb{R}^d$ corresponds to some zero pattern of the polynomials $P_{12}, \ldots, P_{n-1n}$.

Planar distance graphs with high girth

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In 2000 P. O'Donnell proved that

For any $k \in \mathbb{N}$ there exists a planar distance graph with the chromatic number equal to four and with girth larger than k.

The *girth* of a graph the length of its shortest cycle.

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It seems reasonable to ask the following question:

Whether there exists a sequence of distance graphs (complete distance graphs) in $\mathbb{R}^d$, $d = 1, 2, \ldots$, with girth greater than l for a fixed $l \geqslant 3$ and, additionally, the chromatic number of the graphs in the sequence grows exponentially with d?

It turns out that in the case of distance graphs the answer is positive:

Theorem 2 (AK, 2012). For any $g \in \mathbb{N}$ there exists a sequence of distance graphs in $\mathbb{R}^d$, $d = 1, 2, \ldots,$ with girth greater than g such that the chromatic number of the graphs in the sequence grows as $(c + \bar{o}(1))^d$, where $c = c(g) > 1$.

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Proposition 2 (N. Alon, AK, 2013) For any $g \in \mathbb{N}$ there exists a sequence of complete distance graphs in $\mathbb{R}^d,\ d = 1, 2, \dots,$ with girth greater than g such that the chromatic number of the graphs in the sequence grows as $\Omega_g\left(\frac{d}{\log d}\right)$.

Number of edges in distance and complete distance graphs

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It is clear that $L(d) \leqslant l(d) \leqslant \binom{d+2}{2}$, since a complete graph on $d+2$ vertices cannot be realized as a distance graph in $\mathbb{R}^d$.
But is this graph best possible? This question for $l(d)$ was asked by P. Erdős and M. Simonovits in 1980.

Theorem 3

The answer to the question of P. Erdős and M. Simonovits is positive for $d \geqslant 4$:

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Interestingly, this is not the case only for $d = 3$. Graph $K_{3,3}$ is not a distance graph in $\mathbb{R}^3$ and has $9 < \binom{5}{2}$ edges.

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For any $d \geqslant 4$ we have $\binom{d+2}{2} \leqslant L_2(d) \leqslant \binom{d+3}{2} - 6$.

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The graph that gives the upper bound is a bipartite graph K with the parts $A = \{a_1, \ldots, a_d\}, B = \{b_1, \ldots, b_d\}$ and with the set of edges $E = \{(a_i, b_j) : i > j\} \cup \{(a_i, b_j) : i \leqslant 3\}$

Open problems

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