Two notions of unit distance graphs

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Abstract. A *complete (unit) distance graph* in $\mathbb{R}^d$ is a graph whose set of vertices is a finite subset of the d-dimensional Euclidean space, and two vertices are adjacent if and only if the Euclidean distance between them is exactly 1. A *(unit) distance graph* in $\mathbb{R}^d$ is any subgraph of such a graph. We study various properties of both types of distance graphs. We show that for any fixed d the number of complete distance graphs in $\mathbb{R}^d$ on n labelled vertices is $2^{(1+o(1))dn \log_2 n}$, while the number of distance graphs in $\mathbb{R}^d$ on n labelled vertices is $2^{(1-1/\lfloor d/2 \rfloor+o(1))n^2/2}$. This is used to study a Ramsey type question involving these graphs. Finally, we discuss the following problem: what is the minimum number of edges a graph must have so that it is not realizable as a complete distance graph in $\mathbb{R}^d$?

1 Introduction

This paper is devoted to the notion of a (unit) distance graph. There are two well-known definitions:

Definition 1.1. A graph $G = (V, E)$ is a *(unit) distance graph in* $\mathbb{R}^d$ if $V \subset \mathbb{R}^d$ and $E \subseteq \{(x, y), x, y \in \mathbb{R}^d, |x - y| = 1\}$, where $|x - y|$ denotes the Euclidean distance between x and y.

Definition 1.2. A graph $G = (V, E)$ is a *complete (unit) distance graph in* $\mathbb{R}^d$ if $V \subset \mathbb{R}^d$ and $E = \{(x, y), x, y \in \mathbb{R}^d, |x - y| = 1\}$.

Distance graphs arise naturally in the study of two well-known problems of combinatorial geometry. First one, posed by Erdős [3], is the fol-

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lowing: determine the maximum number $f_2(n)$ of unit distances among n points on the plane. Second is the following question, posed by Nelson (see [2]): what is the minimum number $\chi(\mathbb{R}^2)$ of colours needed to color the points of the plane so that no two points at unit distance apart receive the same colour?

Let $\mathcal{D}(d)$ denote the set of all labeled distance graphs in $\mathbb{R}^d$, and let $\mathcal{D}_n(d)$ denote the set of all those of order n. Similarly, denote by $\mathcal{CD}(d)$ the set of all labeled complete distance graphs in $\mathbb{R}^d$, and let $\mathcal{CD}_n(d)$ denote the set of those of order n.

We reformulate the stated above questions in terms of distance graphs:

$$f_2(n) = \max_{G \in \mathcal{D}_n(2)} |E(G)| = \max_{G \in \mathcal{CD}_n(2)} |E(G)|.$$

$$\chi(\mathbb{R}^d) = \max_{G \in \mathcal{D}(d)} \chi(G) = \max_{G \in \mathcal{CD}(d)} \chi(G),$$

The second series of equalities follows from the well-known Erdős– de Bruijn theorem, which states that the chromatic number of the space $\mathbb{R}^d$ is equal to the chromatic number of some finite distance graph in $\mathbb{R}^d$. It is easy to see that it does not matter which definition of distance graph we use in the study of these two problems. However, sets $\mathcal{D}(d)$ and $\mathcal{CD}(d)$ differ greatly, and we discuss it in the next section.

2 Results

The first theorem shows that in any dimension d the number of distance graphs is far bigger that the number of complete distance graphs.

Theorem 2.1.

1. *For any $d \in \mathbb{N}$, we have $|\mathcal{CD}_n(d)| = 2^{(1+o(1))dn \log_2 n}$.*
2. *For any $d \in \mathbb{N}$ we have $|\mathcal{D}_n(d)| = 2^{\left(1-\frac{1}{\lceil d/2 \rceil}+o(1)\right)\frac{n^2}{2}}$.*
3. *If $d = d(n) = o(n)$, then we have $|\mathcal{CD}_n(d)| = 2^{o(n^2)}$.*
4. *If $d = d(n) \geqslant c\frac{n}{\log_2 n}$, where $c > 4$, then $|\mathcal{D}_n(d)| = (1+o(1))2^{\frac{n(n-1)}{2}}$. In other words, almost every graph on n vertices can be realized as a distance graph in $\mathbb{R}^d$.*

Sketch of the proof. Let $P_1, \ldots, P_m$ be m real polynomials in l real variables. For a point $x \in \mathbb{R}^l$ the *zero pattern* of the P_j's at x is the tuple $(\varepsilon_1, \ldots, \varepsilon_m) \in \{0,1\}^m$, where $\varepsilon_j = 0$, if $P_j(x) = 0$ and $\varepsilon_j = 1$ if $P_j(x) \neq 0$. Denote by $z(P_1, \ldots, P_m)$ the total number of zero patterns of polynomials $P_1, \ldots, P_m$. Upper bounds from point 1 and 3 of the theorem are proved using the following proposition from real algebraic geometry ([7], Theorem 1.3 and Corollary 1.5):

Proposition 2.2 ([7]). *Let $P_1, \dots, P_m$ be m real polynomials in l real variables, and suppose the degree of each P_j does not exceed k. Then $z(P_1, \dots, P_m) \leqslant \binom{km}{\ell} \leqslant (ekm/l)^l$.*

Denote by $(v_1^i, \dots, v_d^i)$ the coordinates of the vertex v_i in the distance graph. For each unordered pair $\{i, j\}$ of vertices of the graph define the following polynomial P_{ij}:

$$P_{ij} = -1 + \sum_{r=1}^{d} (v_r^i - v_r^j)^2.$$

It is easy to see that each labeled distance graph in $\mathbb{R}^d$ corresponds to a zero pattern of the polynomials $P_{12}, \dots, P_{n-1n}$.

Lower bound in part 1 of the theorem follows from the fact that certain bipartite graphs with maximum degree d in one part are realizable as complete distance graphs in $\mathbb{R}^d$.

Lower bound in part 1 of the theorem is a corollary of the fact that any $[d/2]$-partite graph is realizable as a distance graph in $\mathbb{R}^d$. Upper bound has several proofs, with the simplest one based on the regularity lemma. It rests on the fact that complete $([d/2]+1)$-partite graph with three vertices in each part is not realizable as distance graph in $\mathbb{R}^d$.

The fourth part of the theorem is an easy consequence of the famous theorem by Bollobás [1] concerning the chromatic number of the random graph $G(n, 1/2)$. □

Next, we study the following two Ramsey-type quantities.

Definition 2.3. The *(complete) distance Ramsey number* $R_D(s, t, d)$ $\big(R_{CD}(s, t, d)\big)$ is the minimum natural m such that for any graph G on m vertices the following holds: either G contains an induced s-vertex subgraph isomorphic to a (complete) distance graph in $\mathbb{R}^d$ or its complement $\bar{G}$ contains an induced t-vertex subgraph isomorphic to a (complete) distance graph in $\mathbb{R}^d$.

The quantity $R_D(s, s, d)$ was introduced in [6], and studied in several follow-up papers, while the quantity $R_{CD}(s, s, d)$ was not studied so far. The following theorem was proved in [4]:

Theorem 2.4 ([4]).

1. *For every fixed $d \geqslant 2$ we have*

$$R_D(s, s, d) \geqslant 2^{\left(\frac{1}{2[d/2]} + o(1)\right)s}.$$

2. *For any $d = d(s)$, $2 \leqslant d \leqslant s/2$ we have*

$$R_D(s, s, d) \leqslant d \cdot R\left(\left\lceil \frac{s}{[d/2]} \right\rceil, \left\lceil \frac{s}{[d/2]} \right\rceil\right),$$

where $R(k, \ell)$ is the classical Ramsey number: the minimum number n so that any graph on n vertices contains either a clique of size k or an independent set of size ℓ.

By the last theorem the bounds for $R_D(s, s, d)$ are roughly the same as for the classical Ramsey number $R\left(\left\lceil \frac{s}{[d/2]} \right\rceil, \left\lceil \frac{s}{[d/2]} \right\rceil\right)$:

$$\frac{s}{2[d/2]}(1 + o(1)) \leqslant \log R_D(s, s, d) \leqslant \frac{2s}{[d/2]}(1 + o(1)),$$

where the $o(1)$-terms tend to zero as s tends to infinity.

Using Theorem 2.1 we can show that $R_{CD}(s, s, d)$ is far larger than $R_D(s, s, d)$.

Theorem 2.5.

1. *For any $d = d(s) = o(s)$ we have $R_{CD}(s, s, d) \geqslant 2^{(1+o(1))s/2}$.*
2. *For $d = d(s) \leqslant cs$, where $c < 1/2$ and $H(c) < 1/2$, there exists a constant $\alpha = \alpha(c) > 0$ such that $R_{CD}(s, s, d) \geqslant 2^{(1+o(1))\alpha s}$.*

This theorem is proved via standard probabilistic approach used to obtain lower bounds on the Ramsey numbers.

The last (possible) difference between $\mathcal{D}(d)$ and $\mathcal{CD}(d)$ we point out is the following. Fix an $l \in \mathbb{N}$. The following theorem was proved in [5].

Theorem 2.6 ([5]). *For any $g \in \mathbb{N}$ there exists a sequence of distance graphs in $\mathbb{R}^d$, $d = 1, 2, \ldots,$ with girth greater than g such that the chromatic number of the graphs in the sequence grows exponentially with d.*

Unfortunately, we cannot prove a similar theorem for complete distance graphs. All we can prove is the following

Proposition 2.7. *For any $g \in \mathbb{N}$ there exists a sequence of complete distance graphs in $\mathbb{R}^d$, $d = 1, 2, \ldots,$ with girth greater than g such that the chromatic number of the graphs in the sequence grows as $\Omega\left(\frac{d}{\log d}\right)$.*

Every bipartite graph is realizable as a distance graph in R^4. However, for any fixed d there exists a bipartite graph that is not realizable as a complete distance graph in $\mathbb{R}^d$. In general, it seems difficult even for a bipartite graph G to decide whether G is realizable as a complete distance graph in $\mathbb{R}^d$ or not. In particular, we introduce the quantity $g_2(d)$, which is equal to the minimum possible number of edges in a bipartite graph K that is not realizable as a complete distance graph in $\mathbb{R}^d$. We obtained the following theorem.

Theorem 2.8. *For any $d \geqslant 4$ we have* $\binom{d+2}{2} \leqslant g_2(d) \leqslant \binom{d+3}{2} - 6$.

The lower bound in this theorem states that if a bipartite graph is not realizable as a complete distance graph in $\mathbb{R}^d$, $d \geqslant 4$, then it must have at least as many edges as the complete graph K_{d+2} on $d+2$ vertices, which is an obvious example of a graph that is not realizable as a distance graph in $\mathbb{R}^d$. This is not the case for $d = 3$, since the graph $K_{3,3}$ is not realizable as a distance graph in $\mathbb{R}^3$ and it has $9 < \binom{5}{2}$ edges. It is interesting to determine, whether an arbitrary graph G that is not realizable as a complete distance graph in $\mathbb{R}^d$, $d \geqslant 4$, must have at least $\binom{d+2}{2}$ edges, and to study the similar question for distance graphs.

Sketch of the proof of theorem 2.8. Both upper and lower bounds are based on linear-algebraic considerations, mostly dealing with the notion of affine dependence.

To obtain the upper bound, we prove that the bipartite graph K'' with the parts $A = \{a_1, \dots, a_d\}$, $B = \{b_1, \dots, b_d\}$ and with the set of edges $E = \{(a_i, b_j) : i > j\} \cup \{(a_i, b_j) : i \leqslant 3\}$ is not realizable as a complete distance graph in $\mathbb{R}^d$.

To prove the lower bound we provide sufficient conditions for a bipartite graph to be realizable as a complete distance graph in $\mathbb{R}^d$. The construction of the realization is algorithmic. □

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