Distance graphs with big girth and large chromatic number

Andrey Kupavskii

Department of Mechanics and Mathematics
Moscow State University

09.09.2012 – 14.09.2012, Nový Smokovec

Nelson-Hadwiger problem

- The following question was asked by E. Nelson in 1950:

the chromatic number

what is the minimum number of colors needed to color the points of the plane so that no two points at unit distance apart receive the same color?

Nelson-Hadwiger problem

- The following question was asked by E. Nelson in 1950:

the chromatic number

what is the minimum number of colors needed to color the points of the plane so that no two points at unit distance apart receive the same color?

The described quantity is called *the chromatic number* $\chi(\mathbb{R}^2)$ of the plane.

Nelson-Hadwiger problem

- The following question was asked by E. Nelson in 1950:

the chromatic number

what is the minimum number of colors needed to color the points of the plane so that no two points at unit distance apart receive the same color?

The described quantity is called *the chromatic number* $\chi(\mathbb{R}^2)$ of the plane.

- We can define analogous quantity in $\mathbb{R}^d$.
- Formally,

$$\chi(\mathbb{R}^d) = \min\{m \in \mathbb{N} : \mathbb{R}^d = H_1 \cup \ldots \cup H_m : \\ \forall i, \forall x, y \in H_i \quad |x - y| \neq 1\}.$$

Distance graphs

definition

We say that a graph $G = (V, E)$ is the *distance graph in* $\mathbb{R}^d$ if $V \subset \mathbb{R}^d$ and $E \subseteq \{(x, y), x, y \in \mathbb{R}^d, |x - y| = 1\}$.

Distance graphs

definition

We say that a graph $G = (V, E)$ is the *distance graph in* $\mathbb{R}^d$ if $V \subset \mathbb{R}^d$ and $E \subseteq \{(x, y), x, y \in \mathbb{R}^d, |x - y| = 1\}$.

1951, P. Erdős, N.G. de Bruijn: If we accept the axiom of choise then the chromatic number of the space $\mathbb{R}^d$ is equal to the chromatic number of some *finite* distance graph in $\mathbb{R}^d$.

Graphs having big girth and large chromatic number

The *girth* of a graph the length of its shortest cycle.
1959, P. Erdős: For any $l, k \in \mathbb{N}$ there exists a graph with chromatic number greater than l and with girth greater than k.

Graphs having big girth and large chromatic number

The *girth* of a graph the length of its shortest cycle.
1959, P. Erdős: For any $l, k \in \mathbb{N}$ there exists a graph with chromatic number greater than l and with girth greater than k.
1968, L. Lovász: explicit construction of such graphs.

Graphs having big girth and large chromatic number

The *girth* of a graph the length of its shortest cycle.
1959, P. Erdős: For any $l, k \in \mathbb{N}$ there exists a graph with chromatic number greater than l and with girth greater than k.
1968, L. Lovász: explicit construction of such graphs.
Question: can we prove results of these type for distance graphs?

Planar distance graphs

It is known that $4 \le \chi(R^2) \le 7$.

Planar distance graphs

It is known that $4 \leq \chi(R^2) \leq 7$.

1975, P. Erdős

Can one construct distance graphs on the plane with chromatic number four and without triangles?

Planar distance graphs

It is known that $4 \leq \chi(R^2) \leq 7$.

1975, P. Erdős

Can one construct distance graphs on the plane with chromatic number four and without triangles?

The answer is yes. Moreover, for any $k \in \mathbb{N}$ there exists a planar distance graph with the chromatic number equal to four and with girth larger than k (P. O'Donnell).

Distance graphs in higher dimensions

It is known that the chromatic number of the space grows exponentially with the dimension:

$(\zeta_{low} + o(1))^n \leq \chi(\mathbb{R}^n) \leq (3 + o(1))^n$, where $\zeta_{low} = 1.239\ldots$ (A. Raigorodskii, D.G. Larman, C.A. Rogers)

Distance graphs in higher dimensions

It is known that the chromatic number of the space grows exponentially with the dimension:

$(\zeta_{low} + o(1))^n \le \chi(\mathbb{R}^n) \le (3 + o(1))^n$, where $\zeta_{low} = 1.239\ldots$ (A. Raigorodskii, D.G. Larman, C.A. Rogers)

Whether there exists a sequence of distance graphs in $\mathbb{R}^d$, $d = 1, 2, \ldots$, such that none of the graphs contain cliques of fixed size, and, additionally, the chromatic number of the graphs in the sequence grows exponentially with d?

Distance graphs in higher dimensions

It is known that the chromatic number of the space grows exponentially with the dimension:

$(\zeta_{low} + o(1))^n \leq \chi(\mathbb{R}^n) \leq (3 + o(1))^n$, where $\zeta_{low} = 1.239\ldots$ (A. Raigorodskii, D.G. Larman, C.A. Rogers)

Whether there exists a sequence of distance graphs in $\mathbb{R}^d$, $d = 1, 2, \ldots$, such that none of the graphs contain cliques of fixed size, and, additionally, the chromatic number of the graphs in the sequence grows exponentially with d?

Analogous question for graphs without odd cycles of length $\leq 2l + 1$ for some fixed l and for graphs with girth $> l$ for some fixed l.

Formulation of the question

We consider the following three families of distance graphs in $\mathbb{R}^n$:
$\mathcal{C}(n,k)$, $\mathcal{G}_{odd}(n,k)$, $\mathcal{G}(n,k)$ are the families of all distance graphs that do not contain complete subgraphs of size k, odd cycles of length $\leq k$ and cycles of length $\leq k$ respectively.

Formulation of the question

We consider the following three families of distance graphs in $\mathbb{R}^n$:
$\mathcal{C}(n,k)$, $\mathcal{G}_{odd}(n,k)$, $\mathcal{G}(n,k)$ are the families of all distance graphs that do not contain complete subgraphs of size k, odd cycles of length $\leq k$ and cycles of length $\leq k$ respectively.
Next we define the following quantities:

$$\zeta_k = \liminf_{n\to\infty} \max_{G\in\mathcal{C}(n,k)} (\chi(G))^{1/n}, \quad \xi_k^{odd} = \liminf_{n\to\infty} \max_{G\in\mathcal{G}_{odd}(n,k)} (\chi(G))^{1/n},$$

$$\xi_k = \liminf_{n\to\infty} \max_{G\in\mathcal{G}(n,k)} (\chi(G))^{1/n}.$$

Formulation of the question

We consider the following three families of distance graphs in $\mathbb{R}^n$:
$\mathcal{C}(n,k)$, $\mathcal{G}_{odd}(n,k)$, $\mathcal{G}(n,k)$ are the families of all distance graphs that do not contain complete subgraphs of size k, odd cycles of length $\le k$ and cycles of length $\le k$ respectively.
Next we define the following quantities:

$$\zeta_k = \liminf_{n\to\infty} \max_{G\in\mathcal{C}(n,k)} (\chi(G))^{1/n}, \quad \xi_k^{odd} = \liminf_{n\to\infty} \max_{G\in\mathcal{G}_{odd}(n,k)} (\chi(G))^{1/n},$$

$$\xi_k = \liminf_{n\to\infty} \max_{G\in\mathcal{G}(n,k)} (\chi(G))^{1/n}.$$

Questions from the previous slide

Whether $\zeta_k > 1$ or not? Same question for ξ_k^{odd}, ξ_k.

Quantities ζ_k, ξ_k^{odd}

Quantities ζ_k, ξ_k^{odd} were studied by A. Raigorodskii, O. Rubanov, E. Demechin and A. Kupavskii.
For the case of ζ_k the positive answer to the question was given by A. Raigorodskii, for the case of ξ_k^{odd} — by E. Demechin, A. Raigorodskii and O. Rubanov.

Quantities ζ_k, ξ_k^{odd}

Quantities ζ_k, ξ_k^{odd} were studied by A. Raigorodskii, O. Rubanov, E. Demechin and A. Kupavskii.
For the case of ζ_k the positive answer to the question was given by A. Raigorodskii, for the case of ξ_k^{odd} — by E. Demechin, A. Raigorodskii and O. Rubanov.
There are two approaches to estimate the quantity ζ_k.

Quantities ζ_k, ξ_k^{odd}

Quantities ζ_k, ξ_k^{odd} were studied by A. Raigorodskii, O. Rubanov, E. Demechin and A. Kupavskii.

For the case of ζ_k the positive answer to the question was given by A. Raigorodskii, for the case of ξ_k^{odd} — by E. Demechin, A. Raigorodskii and O. Rubanov.

There are two approaches to estimate the quantity ζ_k.

Probabilistic (Raigorodskii et al.): we don't obtain an explicit graph, nontrivial bounds for $k \geq 5$ (refinement of this method by Kupavskii works for $k \geq 3$). However, using this technique one can see that $\zeta_k \geq c_k$, where $c_k > 1$ and $\lim_{k \to \infty} c_k = \zeta_{low}$.

Quantities ζ_k, ξ_k^{odd}

Quantities ζ_k, ξ_k^{odd} were studied by A. Raigorodskii, O. Rubanov, E. Demechin and A. Kupavskii.
For the case of ζ_k the positive answer to the question was given by A. Raigorodskii, for the case of ξ_k^{odd} — by E. Demechin, A. Raigorodskii and O. Rubanov.
There are two approaches to estimate the quantity ζ_k.
Probabilistic (Raigorodskii et al.): we don't obtain an explicit graph, nontrivial bounds for $k \geq 5$ (refinement of this method by Kupavskii works for $k \geq 3$). However, using this technique one can see that $\zeta_k \geq c_k$, where $c_k > 1$ and $\lim_{k\to\infty} c_k = \zeta_{low}$.
Code-theoretic: explicit constructions, works for $k \geq 3$, better bounds for small k. But as k grows, the bounds tend to some constant that is significantly smaller than ζ_{low}.
It is also used to prove $\xi_k^{odd} > 1$.

Main result

The main result of this work is the following

Theorem

For any $k \in \mathbb{N}$ there exists a sequence of distance graphs in $\mathbb{R}^d$, $d = 1, 2, \ldots,$ with girth greater than k such that the chromatic number of graphs in the sequence grows exponentially with d.

Main result

The main result of this work is the following

Theorem

For any $k \in \mathbb{N}$ there exists a sequence of distance graphs in $\mathbb{R}^d$, $d = 1, 2, \ldots$, with girth greater than k such that the chromatic number of graphs in the sequence grows exponentially with d.
Another formulation: for any $k \in \mathbb{N}$ we have $\xi_k > 1$.

Main result

The main result of this work is the following

Theorem

For any $k \in \mathbb{N}$ there exists a sequence of distance graphs in $\mathbb{R}^d$, $d = 1, 2, \ldots,$ with girth greater than k such that the chromatic number of graphs in the sequence grows exponentially with d.
Another formulation: for any $k \in \mathbb{N}$ we have $\xi_k > 1$.

Question

Can we provide an explicit construction of such graphs?

The sketch of the proof. Part 1

The proof of the theorem is based on the analysis of the properties of the random subgraphs of the distance graphs $G_{4n} = (V_{4n}, E_{4n})$, where

$$V_{4n} = \{\mathbf{x} = (x_1, \ldots, x_{4n}) : \ x_i \in \{0, 1\}, \ x_1 + \ldots + x_{4n} = 2n\},$$

$$E_{4n} = \{\{\mathbf{x}, \mathbf{y}\} : \ (\mathbf{x}, \mathbf{y}) = n\}.$$

By $(,)$ we denote the scalar product.

The sketch of the proof. Part 1

The proof of the theorem is based on the analysis of the properties of the random subgraphs of the distance graphs $G_{4n} = (V_{4n}, E_{4n})$, where

$$V_{4n} = \{\mathbf{x} = (x_1, \ldots, x_{4n}) : \; x_i \in \{0, 1\}, \; x_1 + \ldots + x_{4n} = 2n\},$$

$$E_{4n} = \{\{\mathbf{x}, \mathbf{y}\} : \; (\mathbf{x}, \mathbf{y}) = n\}.$$

By $(,)$ we denote the scalar product.
Graphs like G_{4n} are used to obtain lower bounds on the chromatic number of the space.

The sketch of the proof. Part 2

It is easy to see that $|V_{4n}| = (2 + o(1))^{4n}, |E_{4n}| = (4 + o(1))^{4n}$.

The sketch of the proof. Part 2

It is easy to see that $|V_{4n}| = (2 + o(1))^{4n}, |E_{4n}| = (4 + o(1))^{4n}$.
One of the main ingridients of the proof is the theorem by P. Frankl and V. Rödl concerning graphs G_{4n}:

Theorem

For any $\epsilon > 0$ there exists $\delta > 0$ such that for any subset S of V_{4n}, $|S| \geq (2-\delta)^{4n}$, the number of edges in S (the cardinality of $E_{4n}|_S$) is greater than $(4-\epsilon)^{4n}$.

The sketch of the proof. Part 3

Lovász local lemma

Let $A_1, \ldots, A_m$ be events in an arbitrary probability space and $J(1), \ldots, J(m)$ be subsets of $\{1, \ldots, m\}$. Suppose there are real numbers γ_i such that $0 < \gamma_i < 1,\ i = 1, \ldots, m$. Suppose the following conditions hold:

- A_i is independent of algebra generated by $\{A_j, j \notin J(i) \cup \{i\}\}$.
- $\mathrm{P}(A_i) \leq \gamma_i \prod_{j \in J(i)} (1 - \gamma_j)$.

Then $\mathrm{P}\left(\bigwedge_{i=1}^m \overline{A_i}\right) \geq \prod_{i=1}^m (1 - \gamma_i) > 0.$

The sketch of the proof. Part 3

Lovász local lemma

Let $A_1, \ldots, A_m$ be events in an arbitrary probability space and $J(1), \ldots, J(m)$ be subsets of $\{1, \ldots, m\}$. Suppose there are real numbers γ_i such that $0 < \gamma_i < 1, \ i = 1, \ldots, m$. Suppose the following conditions hold:

- A_i is independent of algebra generated by $\{A_j, j \notin J(i) \cup \{i\}\}$.
- $\mathrm{P}(A_i) \leq \gamma_i \prod_{j \in J(i)} (1 - \gamma_j)$.

Then $\mathrm{P}\left(\bigwedge_{i=1}^{m} \overline{A_i}\right) \geq \prod_{i=1}^{m} (1 - \gamma_i) > 0$.

Using local lemma we prove that random subgraph of G_{4n} with positive probability does not contain cycles of length less than k and simultaneously the size of maximum independent set in the subgraph is not bigger than $(2 - \epsilon)^{4n}$ for some $\epsilon > 0$.

Thank you for the attention!

Thank you!