On some geometric Ramsey theory problem

Andrey Kupavskii, Maria Titova

Lomonosov Moscow State University,
Moscow Institute of Physics and Technology,
Yandex Research laboratories,
Moscow, Russia

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Classical Ramsey number

Definition

For given $s, t \in \mathbb{N}$ *Ramsey number* $R(s,t)$ is the minimum natural m such that for any graph $G = (V, E)$ on m vertices the following holds: either G contains an independent set of size s or its complement $\bar{G}$ contains a independent set of size t.

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- $R(s,s) \leqslant s^{-c \log s / \log\log s} 4^s$ (D. Conlon, 2009)
- $R(s,s) > \frac{\sqrt{2}}{e}(1+o(1))\, s\, 2^{\frac{s}{2}}$ (J. Spencer, 1975)

Nelson-Hadwiger problem

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the chromatic number

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- We can define analogous quantity in $\mathbb{R}^d$.
- Formally,

$$\chi(\mathbb{R}^d) = \min\{m \in \mathbb{N} : \mathbb{R}^d = H_1 \cup \ldots \cup H_m : \forall i, \forall x, y \in H_i \quad |x - y| \neq 1\}.$$

Distance graphs

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We say that a graph $G = (V, E)$ is the *distance graph in* $\mathbb{R}^d$ if $V \subset \mathbb{R}^d$ and $E \subseteq \{(x, y), x, y \in \mathbb{R}^d, |x - y| = 1\}$.

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1951, P. Erdős, N.G. de Bruijn: If we accept the axiom of choice then the chromatic number of the space $\mathbb{R}^d$ is equal to the chromatic number of some *finite* distance graph in $\mathbb{R}^d$.

Distance graphs. Another motivation

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In any set of n points on the plane there are at most n pairs of points forming a diameter.

This equivalent to the following statement: Let $G = (V, E)$ be a distance graph on the plane, whose vertices lie at distance $\leqslant 1$ apart from each other. Then $\max_G |E| \leqslant n$, if $|V| = n$.

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Distance Ramsey number $R_{\mathrm{NEH}}(s,t,d)$ is the minimum natural m such that for any graph G on m vertices the following holds: either G contains an induced s-vertex subgraph isomorphic to a distance graph in $\mathbb{R}^d$ or its complement $\bar{G}$ contains an induced t-vertex subgraph isomorphic to a distance graph in $\mathbb{R}^d$.

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Best bounds for $d = 2, 3$

Theorem 1 (A. Raigorodskii, M. Titova, 2011)

For $d = 2, 3$ we have

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To prove this theorem they used the following

Claim 1

Fix $d \in \{2, 3\}$. There exist constants $c, \epsilon > 0$, such that for every distance graph $G = (V, E)$ in $\mathbb{R}^d$ of order n we have $|E| \leqslant cn^{2-\epsilon}$.

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This claim, in turn, follows from well-known Kővári-Sós-Turán theorem and the following statement:

If G is a distance graph in $\mathbb{R}^2$ ($\mathbb{R}^3$), then it does not contain $K_{2,3}$ ($K_{3,3}$) as a subgraph.

Best known bounds for $d = 4, \ldots, 8$

Theorem 2 (A. Kupavskii, A. Raigorodskii, M. Titova, 2012)

For $d = 4, \ldots, 8$ the following inequalities hold:

$$R_{\mathrm{NEH}}(s, s, d) \geqslant 2^{\frac{[c_d s]}{2}(1+\bar{o}(1))}.$$

Here

$$c_4 = 0.04413, \qquad c_5 = 0.01833, \qquad c_6 = 0.00806,$$
$$c_7 = 0.00352, \qquad c_8 = 0.00165.$$

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To prove this, we used the following

Claim 2

Fix some $d \in \{4, \ldots, 8\}$. Every distance graph in $\mathbb{R}^d$ of order m has 2^d disjoint independent sets whose total size is at least $[c_d m]$.

New results

Theorem 3

1. (A. Kupavskii, M. Titova) For every fixed $d \geqslant 2$ we have

$$R_{\mathrm{NEH}}(s, s, d) \geqslant 2^{\left(\frac{1}{2[d/2]}+\bar{o}(1)\right)s}.$$

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The first part of Theorem 3 generalizes Theorem 1 for the case $d \geqslant 4$ and significantly improves the bound of Theorem 2. Even for $d = 4$ the bound improves from $2^{0.022\ldots s(1+\bar{o}(1))}$ to $2^{0.25s(1+\bar{o}(1))}$.

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Now the bounds for $R_{\mathrm{NEH}}(s,s,d)$ are roughly the same as for $R\left(\frac{s}{[d/2]}, \frac{s}{[d/2]}\right)$:

$$\frac{s}{2[d/2]}(1+\bar{o}(1)) \leqslant \log R\left(\frac{s}{[d/2]}, \frac{s}{[d/2]}\right), \log R_{\mathrm{NEH}}(s,s,d) \leqslant \frac{2s}{[d/2]}(1+\bar{o}(1))$$

Upper bound

Consider a graph G on $2[d/2] \cdot R\left(\frac{s}{[d/2]}, \frac{s}{[d/2]}\right)$ vertices. Split the set of vertices into $2[d/2]$ equal parts $P_1, \ldots, P_{2[d/2]}$.

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In each part P_i one can find an independent set of size $\geqslant \frac{s}{[d/2]}$ either in $G|_{P_i}$ or in $\bar{G}|_{P_i}$. Suppose we found $\geqslant [d/2]$ disjoint independent sets in induced subgraphs of G. Denote the union of these independent sets by W. Note that $|W| \geqslant s$.

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Then the subgraph $G|_W$ can be realized as a distance graph in $\mathbb{R}^d$, since

Claim 3

For any $l_1, \ldots, l_{[d/2]} \in \mathbb{N}$ any subgraph of $K_{l_1,\ldots,l_{[d/2]}}$ can be realized as a distance graph in $\mathbb{R}^d$.

Lower bound. Auxiliary theorem

Put $k = [d/2] + 1$. Denote by $Cl(G, r)$ the set of r-cliques in graph G and set $cl(G, r) = |Cl(G, r)|$.

Theorem 4

For any fixed $d \geqslant 2$ there exist constants $c, \varepsilon > 0$ such that for any distance graph G in $\mathbb{R}^d$ on n vertices

$$cl\,(G, k) \leqslant cn^{k-\varepsilon}.$$

This theorem is a generalization of Claim 1. To prove it we need the following

Claim 4

Graph $K_{\underbrace{3,\ldots,3}_{k}}$ can not be realized as a distance graph in $\mathbb{R}^d$.

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Claim 5 (corollary from the theorem by P. Erdős, 1964)

For any $l, r \in \mathbb{N}$ there exist constants $c, \varepsilon > 0$ such that any graph G of order n with $cl(G, r) > cn^{r-\varepsilon}$ contains subgraph isomorphic to $K_{\underbrace{l, \ldots, l}_{r}}$.

Lower bound. General idea

To prove the bound $R_{\text{NEH}}(s, s, d) > n$ we need to show that there exists a graph G of order n such that for any s-element subset W of its vertices neither $G|_W$ nor $\bar{G}|_W$ can be realized as a distance graph in $\mathbb{R}^d$.

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We consider random graph model $G(n, 1/2)$. For any s-element subset S of vertices we define event A_S: graph $G(n, 1/2)|_S$ has fewer than $cs^{k-\varepsilon}$ k-cliques.
We define analogous event A'_S for graph $\bar{G}(n, 1/2)|_S$.

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We define analogous event A'_S for graph $\bar{G}(n,1/2)|_S$.
We will show that $\Pr\left(\overline{\bigcup_{S\subset V_n}(A_S \cup A'_S)}\right) > 0$.

Lower bound. Estimation of probability $\Pr(A_S)$.

For $l, k \in \mathbb{N}$ we denote by $\mathcal{P}(l,k)$ the probability that graph $G(l, 1/2)$ does not contain k-cliques.

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We have $\Pr(A_S), \Pr(A'_S) \leqslant (\mathcal{P}(l,k))^{\frac{s^2}{l(l-1)}(1+\bar{o}(1))}$.

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Corollary from the theorem by Rödl, 1985

For fixed l we have $|Sis(S,l)| \sim \frac{s^2}{l(l-1)}$ as $s \to \infty$.

Let σ be a random permutation of $S = \{1, \dots, s\}$. Consider a random variable $F_l^k(\sigma, H)$ which is equal to the number of k-cliques of the graph $\sigma(H)$ that are contained as a whole in one of the l-element sets from the system $Sis(S, l)$.

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Put $G = G(n, 1/2)$. Finally we can estimate the probability of A_S :

$$P(A_S) = P\left(cl(\sigma(G|_S), k) \leqslant cs^{k-\varepsilon}\right) \leqslant P\left(\bigcup_{\sigma}\left(F_k^l\left(\sigma, G|_S\right) \leqslant c' s^{2-\varepsilon}\right)\right).$$

Lower bound. Finishing the proof.

Theorem (P. Erdős, D.J. Kleitman, B.L. Rothschild, 1976)

For any $k \in \mathbb{N}$ we have

$$\mathcal{P}(l,k) \leqslant 2^{-\frac{l^2}{2(k-1)}+\bar{o}(l^2)}.$$

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For any $k \in \mathbb{N}$ we have

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All we have to do is to estimate the probability of $\bigcup_{S \subset V_n} (A_S \cup A'_S)$ by sum of the probabilities of A_S, A'_S

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Analogous problem for distance graphs with all possible edges drawn.
Remark: My conjecture is that for such class of graphs for any fixed d we have

$$R_{\mathrm{NEH}}(s, s, d) \geqslant 2^{\frac{s}{2}(1+\bar{o}(1))}$$

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