Colorings of uniform hypergraphs with large girth

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Introduction

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Theorem (A. Johanssen, 1996)

There exists a constant $C > 0$ such that for any triangle-free graph G we have

$$\chi(G) \leqslant C \, \frac{\Delta(G)}{\ln \Delta(G)}.$$

Definitions from hypergraph theory

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Question: what is the connection between the chromatic number of an n-uniform hypergraph and its maximum vertex degree?

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Let $n \geqslant 3$, $k \geqslant 2$ and let H be an n-uniform hypergraph, satisfying

$$\Delta(H) \leqslant \frac{1}{e} k^{n-1} n^{-1}.$$

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Corollary

Let H be an n-uniform hypergraph, then

$$\chi(H) \leqslant 3(\Delta(H))^{\frac{1}{n-1}}.$$

This theorem gives the right asymptotic order of growth on $\Delta(H)$ (but not on n).

Improvements of the Erdős–Lovász theorem

Let H be an n-uniform hypergraph.

- J. Radhakrishnan, A. Srinivasan (2000).

$$\text{If} \quad \Delta(H) \leqslant 0,17 \frac{2^n}{\sqrt{n \ln n}}, \quad \text{then} \quad \chi(H) = 2.$$

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- D. Shabanov (2010). If $k \geqslant 3$ and

$$\Delta(H) \leqslant \frac{1}{8} \frac{k^{n-1}}{\sqrt{n}}, \quad \text{then} \quad \chi(H) \leqslant k.$$

Hypergraphs with large girth

Definition

A simple cycle of length k in a hypergraph $H = (V, E)$ is an ordered set $(v_0, e_1, v_1, \ldots, e_k, v_k)$ of k distinct edges $e_1, \ldots, e_k$ and k distinct vertices $v_0, \ldots, v_{k-1}$, $v_k = v_0$, such that $v_{i-1}, v_i \in e_i$ for any $i = 1, \ldots, k$.
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Question: Can we improve Erdős–Lovász theorem for hypergraphs with large girth?

Colorings of simple hypergraphs

Next result obtained by Z. Szabó improves the Erdős–Lovász theorem for simple hypergraphs.

Theorem (Z. Szabó, 1990)

For any $\varepsilon > 0$ and $k \geqslant 2$ there exists $n_0(\varepsilon, k)$ such that for all $n > n_0(\varepsilon, k)$ the following statement holds: if H is an n-uniform simple hypergraph, satisfying

$$\Delta(H) \leqslant k^{n-1} n^{-\varepsilon},$$

then H is k-colorable.

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then H is k-colorable.

This theorem gives a better bound for $\Delta(H)$ than all improvements of the Erdős–Lovász theorem.

But k should be small in comparison with n. For large k we have no improvement.

Johanssen-type theorem for simple hypergraphs

Theorem (A. Frieze, D. Mubayi, 2008)

For any $n \geqslant 3$, there exists $c(n) > 0$ such that for any simple n-uniform hypergraph H the following inequality holds

$$\chi(H) \leqslant c(n) \left(\frac{\Delta(H)}{\ln \Delta(H)} \right)^{\frac{1}{n-1}}.$$

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Corollary

For any $n \geqslant 3$ there exists $c(n) > 0$ such that any simple n-uniform hypergraph H with

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Conclusion: the right order of growth on k is found.
Question: what is the order of $c(n)$?
Answer: $c(n) = O(n^{1-2n})$, i.e. theorem of Frieze and Mubayi improves the classical Erdős–Lovász theorem only for $\ln k = \Omega(n^{2n-2})$.

New result

Theorem (A. Kupavskii, D. Shabanov, 2012)

Let H be an n-uniform hypergraph with $g(H) > 5$. If

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where $c > 0$ is an absolute constant, then H is k-colorable.

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where $c > 0$ is an absolute constant, then H is k-colorable.

We have the following relation between girth, chromatic number and maximum degree.

any $g(H)$	$\Delta(H) \leqslant e^{-4k^2} \left(\frac{n}{\ln n}\right)^{a/(a+1)} \frac{k^{n-1}}{n}$, $a = \lfloor \log_2 k \rfloor$, $k = o(\sqrt{\ln \ln n})$	$\Rightarrow \chi(H) \leqslant k$,
any $g(H)$	$\Delta(H) \leqslant c \cdot k^{n-1} n^{-1/2}$	$\Rightarrow \chi(H) \leqslant k$,
$g(H) > 2$	$\Delta(H) \leqslant c \cdot k^{n-1} n^{-\varepsilon(n)}$, $\varepsilon(n) = \Theta(\sqrt[4]{\frac{\ln k}{\ln n}})$	$\Rightarrow \chi(H) \leqslant k$,
$g(H) > 5$	$\Delta(H) \leqslant c \cdot k^{n-1} (\ln n)^{-1}$	$\Rightarrow \chi(H) \leqslant k$.

Ideas of the proof. Random recoloring method

The proof is based on a method of *random recoloring*. This method was first proposed by J. Beck and then developed by J. Spencer, J. Radhakrishnan and A. Srinivasan, A.V. Kostochka.

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The proof is based on a method of *random recoloring*. This method was first proposed by J. Beck and then developed by J. Spencer, J. Radhakrishnan and A. Srinivasan, A.V. Kostochka.

Let $H = (V, E)$ be a hypergraph with $g(H) > 5$ and

$$\Delta(H) \leqslant c\,\frac{k^{n-1}}{\ln n}.$$

We have to show that H is k-colorable. To prove this we shall construct some random k-coloring and estimate the probability that this coloring is not proper for H. Without loss of generality we assume that $V = \{1, \ldots, M\}$. Our construction consists of two stages.

First stage. Initial coloring. We color all vertices from V randomly and uniformly with k colors, independently from each other. Let us denote the generated random coloring by ξ.

First stage

The obtained coloring ξ can contain monochromatic edges and almost monochromatic edges. An edge $e \in E$ is said to be *almost monochromatic* in ξ if there is a color a such that

$$n - s \leqslant |\{v \in e : \ v \text{ is colored with } a \text{ in } \xi\}| \leqslant n - 1,$$

where $1 \leqslant s < n/2$ is the first parameter of the construction. In this case, the color a is called *dominating* in e.

For every $v \in V$, $a = 1, \ldots, r$, let us use the notations

$$\mathcal{M}(v) = \{e \in E : \ v \in e, \ e \text{ is monochromatic in } \xi\},$$

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During the second stage of the construction we shall try to recolor some vertices from the monochromatic edges, but at the same time we shall forbid almost monochromatic edges to become completely monochromatic.

Second stage. Random recoloring process.

Second stage. Process of random recoloring. Consider the following set of mutually independent random elements (also independent of ξ):

1. $N_1 = (N_1(t), t \geq 0), \ldots, N_M = (N_M(t), t \geq 0)$ — standard Poisson random processes.
2. $\{\eta_v^{(r)} : \ v = 1, \ldots, M; \ r \in \mathbb{N}\}$ — equally distributed random variables taking values $1, 2, \ldots, k$ with equal probability p (second parameter of the construction) and the value 0 with probability $1 - kp$.

For each vertex v and color $a \in \{1, 2, \ldots, k\}$ we define the following random variables:

$$r_v(a) = \min \left\{ r : \ \eta_v^{(r)} = a \right\},$$

$$X_v(a) = \{t : \ N_v(t) = r_v(a)\},$$

i.e. $X_v(a)$ is the time of the $r_v(a)$-th jump of N_v.

Second stage. Random recoloring process.

Process of random recoloring goes as follows. For every vertex v and any $r \in \mathbb{N}$, at the time $T_v(r)$ of r-th jump of N_v we check the following three conditions:

1. There is an edge A, $v \in A$, which is monochromatic in the coloring ξ and none of the vertices of A has changed its initial color up to time $T_v(r)$.
2. The color $\eta_v^{(r)} \notin \{0, \xi_v\}$.
3. The recoloring with color $\eta_v^{(r)}$ is not blocked. We say that the recoloring of the vertex v with color a is *blocked*, if there is an edge B, $v \in B$, such that B was almost monochromatic with dominating color a in ξ and at the moment $X_v(a)$ vertex v is the only vertex in B which is not colored with a.

If all the conditions hold then we recolor v with color $\eta_v^{(r)}$. Otherwise, we do not change its color.

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For vertex v and the moment $t > 0$ we get the random variable $\zeta_v(t)$, corresponding to the color of v in the process at the time $t > 0$.

Further actions

What remains to do?

- We have to analyze the situations in which there are monochromatic edges in the random coloring $\zeta(t) = \{\zeta_1(t), \ldots, \zeta_M(t)\}$. The event $\mathcal{F}(t)$ that $\zeta(t)$ is not a proper coloring for the hypergraph H can be divided into some "local" events, which depend on the colorings of adjacent edges.
- We estimate the probabilities of these local events and make the choice of the parameters (s, p, t).
- Finally, we use Local Lemma to show that all of them do not occur simultaneously with positive probability.

Thank You

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