# The chromatic numbers of distance graphs and applications to combinatorial problems

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# Nelson–Hadwiger problem

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**the chromatic number**

what is the minimum number of colors which are needed to paint all the points on the plane so that any two points at distance 1 apart receive different colors?

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- the same quantity can be considered in $\mathbb{R}^d$.
- Formally,

$$\chi(\mathbb{R}^d) = \min\{m \in \mathbb{N} : \mathbb{R}^d = H_1 \cup \ldots \cup H_m : \\ \forall i, \forall x, y \in H_i \quad |x - y| \neq 1\}.$$

# Distance graph

**definition**

The *distance graph* $G = (V, E)$ in $\mathbb{R}^d$ is a graph with $V \subset \mathbb{R}^d$ and $E = \{(x, y), x, y \in \mathbb{R}^d, |x - y| = 1\}$.

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**Theorem, 1951, Erdős, de Bruijn**

If we accept the axiom of choice, then the chromatic number of the space is equal to the chromatic number of some **finite** distance graph in that space.

# Asymptotical lower bounds

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What can we obtain for distance graphs?

# Distance graphs without cliques and cycles

We know, that $\chi(\mathbb{R}^d) \geq (1,239.. + o(1))^d$, or, that there is a finite distance graph $G$ in $\mathbb{R}^d$ with $\chi(G) \geq (1,239.. + o(1))^d$.

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**So, we want to obtain the results of the form:**

$G$ is a finite distance graph in $\mathbb{R}^d$, $G$ does not contain clique of size $k \geq 3$ (cycle of length $l \geq 3$), and $\chi(G) \geq (c + \bar{o}(1))^d, c > 1$.

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**Theorem, Raigorodskii, Rubanov**

For all $k$ there is a distance graph $G$ in $\mathbb{R}^d$, $G$ does not contain cliques of size $k$, $\chi(G) \geq (c + \bar{o}(1))^d, c > 1$. Moreover, $c \to 1,239..$ as $k \to \infty$.

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We can also obtain an explicit construction of the graph $G$ with $\chi(G) \geq (c + \bar{o}(1))^d, c > 1$ and without **odd** cycles of length $\leq l$. Unfortunately, we can't say anything about even cycles.

# New results

Denote by $\chi_k(\mathbb{R}^d)$ the maximum of the chromatic number among all distance graphs in $\mathbb{R}^d$ that do not contain cliques of size $k$.

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| $n$ | $c, \chi_k(\mathbb{R}^d) \geq (c+\bar{o}(1))^d$ – previous bound | new bound using $(0,1)$-vectors | new bound using $(-1,0,1)$-vectors |
|---|---|---|---|
| 3 | 1.0582 | 1.0582 | – |
| 4 | 1.0582 | **1.0663** | 1.0374 |
| 5 | 1.0582 | **1.0857** | 1.0601 |
| 6 | 1.0743 | **1.0898** | 1.0754 |
| 7 | 1.0857 | **1.0995** | 1.0865 |
| 8 | 1.0933 | **1.1019** | 1.0948 |
| 9 | 1.0992 | **1.1077** | 1.1013 |
| 10 | 1.1033 | **1.1093** | 1.1066 |
| 11 | 1.1075 | **1.1131** | 1.1109 |
| 12 | 1.1096 | 1.1142 | **1.1145** |
| 13 | 1.1124 | 1.1170 | **1.1175** |
| 14 | 1.1151 | 1.1178 | **1.1201** |
| 15 | 1.1220 | 1.1198 | **1.1224** |
| $\lim_{k\to\infty}$ | **1.239** | 1.139 | 1.154 |

# $(0,1)$-graphs, $(-1,0,1)$-graphs.

In fact, all the bounds discussed above are obtained on the graphs of the following type.

$G = G(d, m, \{a_0, a_1, \ldots, a_m\}, x) = (V, E)$. The set of vertices is:

$$V = \{x = (x_1, \ldots, x_d), x_i \in \{0, 1, \ldots, m\},$$

$$|\{i : x_i = j\}| = a_j d,\ \forall j = 0, \ldots, m,\ a_i \in (0,1), \sum_{i=0}^{m} a_i = 1\}.$$

The set of edges is: $E = \{\{y_1, y_2\} | y_1, y_2 \in V, (y_1, y_2) = xd\}$.

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We are mostly interested in cases $m = 1$ and $m = 2$, i.e. in so-called $(0,1)$-graphs and $(-1,0,1)$-graphs.

# threshold for containing a $k$-clique

We want to know for a given set of parameters $a_i, x$ whether the graph $G = G(d, m, \{a_0, a_1, \ldots, a_m\}, x)$ contains cliques of size $k$ or not.

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In general, we obtained the following theorem:

**Theorem**

Consider the graph $G = G(d, m, \{a_0, a_1, \dots, a_m\}, x)$ in the sequence of dimensions $d_1, d_2, \dots,$ such that $d_i a_j, d_i x \in \mathbb{N}\ \forall i, j$. Then if

$$x < \frac{(k\sum_{i=0}^m ia_i)^2 - \{k\sum_{i=0}^m ia_i\}^2 + \{k\sum_{i=0}^m ia_i\} - k\sum_{j=0}^m j^2 a_j}{k(k-1)},$$

then $G$ does not contain cliques of size $k$.

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then $G$ does not contain cliques of size $k$.

In fact, this bound is in some sense sharp.

# threshold for containing a $k$-clique. $(0,1)$ case

**Theorem**

Let $m = 1, a_1 = a, x > 0, a, x \in \mathbb{Q}$, and consider the sequence of dimensions $d_1, d_2, \ldots,$ which satisfies the condition $ad_i, xd_i \in \mathbb{N}$. Consider distance graphs $G_i = G(d_i, 1, \{1-a, a\}, x)$. Let $k$ be a natural number, $k \geq 3$. If

$$x < \frac{(ka)^2 - \{ka\}^2 - [ka]}{k(k-1)} = f_1,$$

then $G_i$ do not contain complete subgraphs (cliques) on $k$ vertices.

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then $G_i$ do not contain complete subgraphs (cliques) on $k$ vertices.
Moreover, this bound is in some sense sharp. Namely, there exists a constant $c = c(k, a)$, such that in the sequence of dimensions $cd_1, cd_2, \ldots$ graphs $\tilde{G}_i = G(cd_i, 1, \{1-a, a\}, f_1)$ contain complete subgraphs on $k$ vertices.

# threshold for containing a $k$-clique. $(-1, 0, 1)$ case

**Theorem**

Consider the graph $G = G(a, b, x) = (V, E)$ with the set of vertices:

$$V = \{x = (x_1, \ldots, x_d), x_i \in \{-1, 0, 1\},$$
$$|\{i : x_i = -1\}| = bd, \ |\{i : x_i = 1\}| = ad, \ a, b \in (0, 1), a + b \leq 1\}$$

and with the set of edges: $E = \{\{y_1, y_2\} | y_1, y_2 \in V, (y_1, y_2) = -xd\}$. If

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$G$ does not contain $k$-cliques.
Moreover, $G$ does not contain $k$-cliques if $\{ka\} + k(1 - a - b) < 1$ and

$$x > \frac{(k-k^2)(2a+2b-1) + (4k-4)\{ka\} + 4\{ka\}^2 + 4k(a+b)[ka] - 4(ka)^2}{k(k-1)}.$$

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These bounds are sharp

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**Lemma.**

If $Q \subset V$ is such that $|Q| > \sum_{i=0}^{p-1} \binom{d}{i}$, then there exist $\mathbf{z}, \mathbf{y} \in Q$ with $(\mathbf{z}, \mathbf{y}) = xd$.

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In other words, $\alpha(G) \le \sum_{i=0}^{p-1}\binom{d}{i}$, and the chromatic number

$$\chi(G) \ge |V|/\alpha(G) \ge \frac{\binom{d}{ad}}{\sum_{i=0}^{p-1}\binom{d}{i}}.$$

# Borsuk partition problem

**The following problem was posed by K. Borsuk in 1933:**

Is it true that any set $\Omega \subset \mathbb{R}^d$ having diameter 1 can be divided into some parts $\Omega_1, \ldots, \Omega_{d+1}$ whose diameters are strictly smaller than 1?

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- $$\operatorname{diam} \Omega = \sup_{\mathbf{x},\mathbf{y} \in \Omega} |\mathbf{x} - \mathbf{y}|$$
- By $f(\Omega)$ we denote the value

$$f(\Omega) = \min\{f : \ \Omega = \Omega_1 \cup \ldots \cup \Omega_f, \ \ \forall\, i \ \ \operatorname{diam} \Omega_i < \operatorname{diam} \Omega\}.$$

$$f(d) = \max_{\Omega \subset \mathbb{R}^d, \ \operatorname{diam} \Omega = 1} f(\Omega).$$

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- Borsuk's problem: *is it true that always* $f(d) = d + 1$*?*

# History and some known results

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2. Borsuk's conjecture is shown to be true for $d \le 3$ and false for $d \ge 298$
3. $(1.2255... + o(1))^{\sqrt{d}} \le f(d) \le (1.224... + o(1))^d$.

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To a finite set of points $\Omega$ with a unit diameter we assign the following graph $G_\Omega = (V, E)$: $V$ consists of all the points of $\Omega$. $E$ consists of all pairs of points $x, y \in \Omega, \|x - y\| = 1$.

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To obtain a counterexample to Borsuk conjecture we need to construct a graph of diameters with big chromatic number, namely, bigger than dimension plus one.

# Graphs of diameters with certain properties

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**Theorem**

For any $r > \frac{1}{2}$, there exists a $d_0 = d_0(r)$ such that for every $d \geq d_0$, one can find a set $\Omega \subset S_r^{d-1}$ which has diameter 1 and does not admit a partition into $d+1$ parts of smaller diameter.

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For any $r > \frac{1}{2}$, there exists a $d_0 = d_0(r)$ such that for every $d \geq d_0$, one can find a set $\Omega \subset S_r^{d-1}$ which has diameter 1 and does not admit a partition into $d+1$ parts of smaller diameter.

In other words, for every $r > 1/2$ there exists $\Omega \subset \mathbb{R}^d$, such that $\chi(G_\Omega) > d+1$, and $G_\Omega$ lies on the sphere of radius $r$.

The last condition on $G$ is very strong, because if $r$ is small enough, then graph $G$ surely does not contain cliques and odd cycles of given size.

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The main question is how to make a graph of diameters out of a distance graph.

**dual mappings of the type**

$$\mathbf{y} = (y_1, \ldots, y_n) \to \mathbf{y}^{*2} = (y_1^2, y_1 y_2, \ldots, y_n y_{n-1}, y_n^2).$$

# The end

Thank You