The chromatic numbers of the normed spaces

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Nelson–Hadwiger problem

- The following problem was posed by Nelson in 1950:

the chromatic number

what is the minimum number of colors which are needed to paint all the points on the plane so that any two points at distance 1 apart receive different colors?

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- Formally,

$$\chi(\mathbb{R}^d) = \min\{m \in \mathbb{N} : \mathbb{R}^d = H_1 \cup \ldots \cup H_m : \\ \forall i, \forall x, y \in H_i \quad |x - y| \neq 1\}.$$

Distance graph

definition

the *distance graph* $G = (V, E)$ *in* $\mathbb{R}^d$ is a graph with $V \subset \mathbb{R}^d$ and $E = \{(x, y), x, y \in \mathbb{R}^d, |x - y| = 1\}$.

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1951, Erdős, de Bruijn: If we accept the axiom of choice, then the chromatic number of the space is equal to the chromatic number of some finite distance graph in that space.

Some known results in small dimensions

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dim	4	5	6	7	8	9	10	11	12
$\chi \geq$	7	9	11	15	16	21	23	25	27

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- 2000, Raigorodskii, $\chi(\mathbb{R}^d) \geq (1,239.. + o(1))^d$

Generalizations and related problems

In the definition of the chromatic number instead of $\mathbb{R}^n$ with Euclidean metric we can consider an arbitrary space with an arbitrary metric.

- There is a large number of results concerning the chromatic number of $\mathbb{Q}^d$ and S^d with Euclidean metric and the chromatic number of the space $\mathbb{R}^d_p$ with l_p-metric.

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We also can consider colorings of the space of certain type.

- *Measurable chromatic number* $\chi^m(\mathbb{R}^2)$ (i.e. each color is a measurable set) is well-studied. We have $5 \leq \chi^m(\mathbb{R}^2) \leq 7$.

Asymptotical bounds

We denote by $\chi(\mathbb{R}^d_K, A)$ the chromatic number of the space with the norm induced by a convex centrally symmetric bounded body K and with the set A of forbidden distances.
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1. For $p = 2$ (classical case) we have $(1,239.. + o(1))^d \leq \chi(\mathbb{R}^d_2) = \chi(\mathbb{R}^d) \leq (3 + o(1))^d$. The upper bound is due to Larman, Rogers, 1972.

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2. We have

$$\chi(\mathbb{R}^d_p) \geq (1,207... + o(1))^d,$$
$$\chi(\mathbb{R}^d_\infty) = 2^d \quad \text{and}$$
$$\chi(\mathbb{R}^d_1) \geq (1,365... + o(1))^d.$$

Last result is due to Raigorodskii.

New results

Füredi and Kang in several works established different upper estimates on the $\chi(\mathbb{R}_p^d)$. The best one is

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Theorem 1

We have

$$\chi(\mathbb{R}_K^d) \leq \frac{(\ln d + \ln\ln d + \ln 4 + 1 + o(1))}{\ln\sqrt{2}} \cdot 4^d.$$

New results. l_p case.

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We have

$$\chi(\mathbb{R}_p^d) \leq 2^{(1+c_p+\delta_d)d},$$

where $\delta_d \to 0$ as $d \to \infty$, and $c_p < 1$ as $p > 2$ and $c_p \to 0$ as $p \to \infty$.

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In particular, for $p(d) > \omega(d) d \ln\ln d$, $\omega(d) \to \infty$, we can obtain

$$\chi(\mathbb{R}^d_{p(d)}) \leq (\ln d + \ln\ln d + \ln 2 + 1 + o(1)) d 2^d = (2 + o(1))^d.$$

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Remind, that $\chi(\mathbb{R}^d_\infty) = 2^d$.

Main ingredients of the proof

1. At first we prove a slight variation of Erdős–Rogers theorem (1962) about covering the space by copies of convex bodies:
   With some conditions on the structure of the set X the following theorem holds:

Theorem (Erdős, Rogers)

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4. Then we construct a suitable lattice packing and cover the space by its translates using the covering technique from item 1.

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2. Let $l \geq 2$. Then $\chi(\mathbb{R}^d_p, A) \geq (b \cdot l)^d$, where $b = \frac{\sqrt[p']{2}}{2}$ and $p' = \max\{p, \frac{p}{p-1}\}$.

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3. Let $l \ge 2$. Then $\chi(\mathbb{R}^d, A) \ge (b \cdot l)^d$ where $b \approx 0,755 \cdot \sqrt{2}$.

Comment on Theorem 3. The chromatic number with multiple forbidden distances

We will limit ourself to the Euclidean case.
Let B be an arbitrary k-element set.

In general, we have an upper bound $\chi(\mathbb{R}^d, B) \le (3 + o(1))^{dk}$.

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$B_0 \subset A = [1, l]$ if $l = \sqrt{k}$, so, by Theorem 3,

$\chi(\mathbb{R}^d, B_0) \le (2(\sqrt{k} + 1) + o(1))^d = (c_1' k)^{c_2' d}$ with some c_1', c_2'.

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Unfortunately, Theorem 3 does not give an improvement of the estimate from item 1 for an arbitrary k-element set B.

Comment on theorem 4. The gap between upper and lower bounds

1. In case of an arbitrary norm the gap between upper and lower bound in theorems 3 and 4 is

$(4\frac{l+1}{l} + \bar{o}(1))^d = (4 + \bar{o}(1))^d$ as $l, d \to \infty$.

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2. In the Euclidean case the gap is equal to

$$(\tfrac{2}{b}^{\frac{l+1}{l}} + \bar{o}(1))^d \approx (1,87^{\frac{l+1}{l}} + \bar{o}(1))^d = (1,87 + \bar{o}(1))^d \text{ as } l, d \to \infty.$$

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3. This is even better than the gap between upper and lower bounds for classical chromatic number. It is equal to

$$(\tfrac{3}{1.239} + \bar{o}(1))^d \approx (2.421 + \bar{o}(1))^d. \text{ as } d \to \infty.$$

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2. The technique used to obtain lower bounds is also based on a construction of some packing.
3. Additional ingredients are famous Kabatyanskiy – Levenshtein bound and Pichugov's bound on the radius of Jung's ball in $\mathbb{R}_p^d$.

The notion of $m_1(\mathbb{R}^n)$

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Definition

The *extreme density* of such set is

$$m_1(\mathbb{R}^n) = \sup \{\delta(A) : A \subseteq \mathbb{R}^n \text{ is measurable and avoids unit distance}\}.$$

The extreme density $m_1(\mathbb{R}^n)$: motivation

Relation between $m_1(\mathbb{R}^n)$ and the *measurable chromatic number* $\chi^m(\mathbb{R}^n)$ of the Euclidean space:

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The *chromatic number* $\chi(\mathbb{R}^n)$ is the minimum number of colors needed to paint all the points in $\mathbb{R}^n$ in such a way that any two points at unit distance apart receive different colors.

For $\chi^m(\mathbb{R}^n)$ it is additionally required that points receiving the same color form Lebesgue measurable sets.

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$\chi^m(\mathbb{R}^n) \geq 1/m_1(\mathbb{R}^n) \Longrightarrow$

upper bounds on $m_1(\mathbb{R}^n)$ are lower bounds on the measurable chromatic number $\chi^m(\mathbb{R}^n)$

Bounds on $\chi^m(\mathbb{R}^n)$ and $m_1(\mathbb{R}^n)$

Upper bounds on $m_1(\mathbb{R}^n), n \geq 2$, are due to F. M. de Oliveira Filho, F. Vallentin (2008). Bound on $\chi^m(\mathbb{R}^2)$ is due to K.J. Falconer (1981). The only case where lower bound on $\chi^m(\mathbb{R}^n)$ is better than $1/m_1(\mathbb{R}^n)$ is the case of the plane.

n	$\chi^m(\mathbb{R}^n) \geq$	$m_1(\mathbb{R}^n) \leq$
2	5	0.26841
3	7	0.16560
4	9	0.11293
5	14	0.07528
6	20	0.05157
7	28	0.03612
8	39	0.02579
9	54	0.01873
10	73	0.01380
11	97	0.01031
12	129	0.00780

Sets avoiding unit distance: motivation

- Study of $\chi(\mathbb{R}^n)$:

With some conditions on the structure of the set X the following theorem holds:

Theorem (Erdős, Rogers)

There exists a covering of the space by $\delta^{-1}(X) \cdot n \ln n(1 + o(1))$ copies of the set X.

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Corollary

If X is a set avoiding unit distance, then holds

$$\chi(\mathbb{R}^n) \leq \delta^{-1}(X) \cdot n \ln n(1 + o(1)).$$

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$$\chi(\mathbb{R}^n) \leq \delta^{-1}(X) \cdot n \ln n(1 + o(1)).$$

The best known asymptotic upper bound on the chromatic number of $\mathbb{R}^n$ is obtained using this theorem: $\chi(\mathbb{R}^n) \leq (3 + o(1))^n$.

Lower bounds on $m_1(\mathbb{R}^n)$: main result

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2. We obtain new lower bounds on $m_1(\mathbb{R}^n)$, $n = 3, \ldots, 8$.

Theorem (Kupavskii, Titova)

The following inequalities hold:

$$\begin{aligned} m_1(\mathbb{R}^3) &\geq 0.09877, & m_1(\mathbb{R}^6) &\geq 0.00806, \\ m_1(\mathbb{R}^4) &\geq 0.04413, & m_1(\mathbb{R}^7) &\geq 0.00352, \\ m_1(\mathbb{R}^5) &\geq 0.01833, & m_1(\mathbb{R}^8) &\geq 0.00165. \end{aligned}$$

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Theorem (Kupavskii, Titova)

The following inequalities hold:

$$\begin{aligned} m_1(\mathbb{R}^3) &\geq 0.09877, & m_1(\mathbb{R}^6) &\geq 0.00806, \\ m_1(\mathbb{R}^4) &\geq 0.04413, & m_1(\mathbb{R}^7) &\geq 0.00352, \\ m_1(\mathbb{R}^5) &\geq 0.01833, & m_1(\mathbb{R}^8) &\geq 0.00165. \end{aligned}$$

Note: sets constructed to obtain these bounds satisfy the conditions of Erdős – Rogers theorem.

Comparison of the results

n	the biggest known density of the packing	known upper bound $m_1(\mathbb{R}^n)$ (Filho, Vallentin)	known lower bound $m_1(\mathbb{R}^n)$	new lower bound $m_1(\mathbb{R}^n)$
2	0.90689	0.26841	0.2293 (Croft)	—
3	0.74048	0.16560	0.09256	0.09877
4	0.61685	0.11293	0.03855	0.04413
5	0.46526	0.07528	0.01453	0.01833
6	0.37295	0.05157	0.00582	0.00806
7	0.29530	0.03612	0.00230	0.00352
8	0.25367	0.02579	0.00099	0.00165

The end

Thank You