Sbornik: Mathematics **202**:6 859–886

Matematicheskiĭ Sbornik **202**:6 83–110 DOI 10.1070/SM2011v202n06ABEH004169

On the colouring of spheres embedded in $\mathbb{R}^n$

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Abstract. The work concerns the well-known problem of identifying the chromatic number $\chi(\mathbb{R}^n)$ of the space $\mathbb{R}^n$, that is, finding the minimal number of colours required to colour all points of the space in such a way that any two points at distance one from each other have different colours. A new quantity generalising the chromatic number is introduced in the paper, namely, the speckledness of a subset in a fixed metric space. A series of lower bounds for the speckledness of spheres is derived. These bounds are used to obtain general results lifting lower bounds for the chromatic number of a space to higher dimensions, generalising the well-known 'Moser-Raisky spindle'. As a corollary of these results, the best known lower bound for the chromatic number $\chi(\mathbb{R}^{12}) \geqslant 27$ is obtained, and furthermore, the known bound $\chi(\mathbb{R}^4) \geqslant 7$ is reproved in several different ways.

Bibliography: 23 titles.

Keywords: chromatic number, distance graph, speckledness of a set.

§ 1. Introduction

In 1950 Hadwiger and Nelson formulated the problem of finding the chromatic number of the plane, that is, the minimal number of colours required to colour all points on the plane in such a way that any two points at distance one from each other have different colours.

This problem can be generalised as follows. Let (Γ,ρ) be a metric space (Γ is a space, and ρ is a metric). A *regular colouring* of the metric space with m colours is defined as the following function F:

$$F = F_m(x)\colon \Gamma \xrightarrow{F} \{1,\dots,m\}, \qquad (F(x_0)=F(x)) \quad \Longrightarrow \quad \rho(x,x_0)\neq 1.$$

The *chromatic number of the space* is the number

$$\chi((\Gamma,\rho)) = \min\{m\in\mathbb{N} : \exists\, F_m\}.$$

The chromatic number of the space $\mathbb{R}^n$ with the Euclidean metric is denoted by $\chi(\mathbb{R}^n)$. We shall write $\chi(\Gamma)$ instead of $\chi((\Gamma,\rho))$ whenever $\Gamma\subset\mathbb{R}^n$ and the metric on Γ is Euclidean.

We note that even the problem of finding the chromatic number of the plane has not been solved. There are simple bounds for this number, namely, $4\leqslant\chi(\mathbb{R}^2)\leqslant 7$, and all attempts to improve these bounds have been unsuccessful.

This research was supported by the Russian Foundation for Basic Research (grant no. 09-01-00294) and grant no. МД-8390.20.10 of the President of the Russian Federation.

AMS 2010 *Mathematics Subject Classification.* Primary 05C15.

On the other hand, during the 60 years since the appearance of the chromatic number problem many results have been obtained in this field, see [1]–[6]. The chromatic numbers have been studied for some special colourings of a Euclidean space (see [4], [5]), and for real spaces with non-Euclidean metrics (see [1]).

Lower bounds are known for the chromatic number of the space $\mathbb{Q}^n$ with the metric obtained by the restriction of the Euclidean metric of $\mathbb{R}^n$ for $n \leqslant 7$ (see [1], [7]). In particular, $\chi(\mathbb{Q}^2) = 2$ (see [8]), $\chi(\mathbb{Q}^3) = 2$ (see [9]) and $\chi(\mathbb{Q}^4) = 4$ (see [10]).

For the chromatic number of the sphere S^2 with the metric obtained by the restriction of the Euclidean metric of $\mathbb{R}^3$, bounds are known which depend on the radius of the sphere (see [11], [12]).

An asymptotic lower bound for the chromatic number of $\mathbb{R}^n$ with the Euclidean metric is proved in [1]; an upper bound for the same number is given in [13]. Thus, it is proved that

$$(1.239 + o(1))^n \leqslant \chi(\mathbb{R}^n) \leqslant (3 + o(1))^n,$$

which implies that the number $\chi(\mathbb{R}^n)$ grows exponentially in n. The exponential growth of the number $\chi(\mathbb{R}^n)$ is also established for any ℓ_p-metric (see [14], [15]).

The following bounds for $n = 3, 4$ are known: $6 \leqslant \chi(\mathbb{R}^3) \leqslant 15$ (see [16], [17]) and $7 \leqslant \chi(\mathbb{R}^4) \leqslant 54$ (see [17], [18]).

Table 1

dim	1	2	3	4
$\chi \geqslant$	2	4 (see [21])	6 (see [16])	7 (see [18], [22])
dim	5	6	7	8
$\chi \geqslant$	9 (see [18])	11 (see [7])	15 (see [1])	16 (see [13])
dim	9	10	11	12
$\chi \geqslant$	21 (see [19])	23 (see [19])	25 (see [20])	25 (see [20])

All known upper bounds for $\chi(\mathbb{R}^n)$ with $n > 4$ are trivial. As for lower bounds, a construction from the work [13] led to bounds in low dimensions which were unimprovable for a long time. Recently, these lower bounds for $\chi(\mathbb{R}^n)$ have been improved in [19] and [20] for $n = 9, \dots, 12$. The known bounds for $\chi(\mathbb{R}^n)$ with $n \leqslant 12$ are given in Table 1, together with references to the papers where they were obtained.

§ 2. Statement of the problem

Let (Γ, ρ) be a metric space, $U \subset \Gamma$, and let ρ_U be the restriction of the metric ρ to U. The *speckledness* $\pi(U|(\Gamma, \rho))$ *of a set* U *with respect to the space* (Γ, ρ) is defined as

$$\pi(U|(\Gamma, \rho)) = \min_F \max_{\mathscr{O}} \chi'(\mathscr{O}(U)),$$

where $\mathscr{O}$ is an arbitrary transformation of the space preserving the metric, F is a regular colouring of the space, and $\chi'(\mathscr{O}(U))$ is the number of colours used to colour the set $\mathscr{O}(U)$ in the given colouring. It is easy to see that

$$\chi((U,\rho_U)) \leqslant \pi(U|(\Gamma,\rho)) \leqslant \pi(U|(\Gamma',\rho'))$$

whenever $\Gamma \subset \Gamma'$, assuming that the metric ρ is obtained by the restriction of the metric ρ' to Γ.

The improvement of the lower bounds for the chromatic number of Euclidean spaces in [16], [20], [22] builds upon a construction of circles of different radii and large speckledness in the space $\mathbb{R}^n$. Our interest in the study of speckledness of spheres is justified by the fact that spheres of large speckledness allow us to lift lower bounds for the chromatic number to higher dimensions.

It is easy to see that $\chi((U,\rho_U)) \neq \pi(U|(\Gamma,\rho))$ in general. For example, taking $\mathbb{R}^1$ as (Γ,ρ), and taking a pair of points at distance $\frac{1}{2}$ from each other as U, we obtain $\chi(U)=1$ and $\pi(U|\mathbb{R}^1)=2$. The latter identity comes from the fact that in a sequence of three points on the line at distance $\frac{1}{2}$ from each other the end points have different colours, and therefore the middle point and one of the end points form a differently coloured pair.

In this paper we are mainly concerned with the speckledness of spheres

$$\pi^n(S^m_r) = \pi(S^m_r|\mathbb{R}^n), \qquad m \leqslant n-1,$$

which depends on the radius r and the dimension m. As a corollary of our results on the speckledness of spheres we obtain new lower bounds for the chromatic numbers of Euclidean spaces.

§ 3. Formulation of the results

3.1. One-dimensional case. In this section we formulate the result on the speckledness of circles.

Theorem 1. 1) *Let* $n \geqslant 4$ *and* $r \in \Bigl(\sqrt{\frac{1}{2+\sqrt{3}}}, \frac{1}{\sqrt{3}}\Bigr) \cup \Bigl(\frac{1}{\sqrt{3}}, \sqrt{2+\sqrt{3}}\Bigr)$. *Then* $\pi^n(S^1_r) \geqslant 4$.

2) *Let* $n \geqslant 5$ *and* $r \in \bigl\{\frac{1}{\sqrt{3}}\bigr\} \cup [\sqrt{2+\sqrt{3}}, \infty)$. *Then* $\pi^n(S^1_r) \geqslant 4$.

3) *Let* $n \geqslant 2$ *and* $r > \frac{1}{2}$. *Then* $\pi^n(S^1_r) \geqslant 3$.

4) *Let* $n \geqslant 3$ *and* $r > \frac{1}{4}$. *Then* $\pi^n(S^1_r) \geqslant 3$.

5) *Let* $n=3$. *Then the union* $\Bigl(\sqrt{\frac{1}{2+\sqrt{3}}}, \frac{1}{\sqrt{3}}\Bigr) \cup \Bigl(\frac{1}{\sqrt{3}}, \sqrt{2+\sqrt{3}}\Bigr)$ *contains a countable dense subset* U *such that for any* $r \in U$ *the inequality* $\pi^3(S^1_r) \geqslant 4$ *holds.*

Note that statement 1) in the theorem above is simply a reformulation of the main result of [20] in our terms.

The proof of statement 5) of Theorem 1 closely follows the proof of statement 1), but will be given separately in a condensed form.

3.2. Two-dimensional case. In this section we consider two-dimensional spheres. Using the methods developed here we shall give a new proof of the bound $\chi(\mathbb{R}^4) \geqslant 7$ and prove the bound $\chi(\mathbb{R}^{12}) \geqslant 27$.

Theorem 2. 1) *For* $n \geqslant 4$ *and* $r > \frac{1}{2\sqrt{\sqrt{3}-1}} \approx 0.584$, $r \neq \sqrt{\frac{3}{8}}$, *the bound* $\pi^n(S^2_r) \geqslant 5$ *holds.*

2) *For* $n \geqslant 5$ *and* $r = \sqrt{\frac{3}{8}}$, *the bound* $\pi^n(S_r^2) \geqslant 5$ *holds.*

3) *For* $n \geqslant 6$ *and* $r > \frac{1}{\sqrt{10}} + \sqrt{\frac{5}{8}}(2-\sqrt{3}) \approx 0.528$, *the bound* $\pi^n(S_r^2) \geqslant 5$ *holds.*

4) *For* $n \geqslant 7$ *and* $r > \frac{1}{\sqrt{2+\sqrt{3}}} \approx 0.518$, *the bound* $\pi^n(S_r^2) \geqslant 5$ *holds.*

5) *Let* $n = 3$. *Then the interval* $\left(\frac{1}{2\sqrt{\sqrt{3}-1}}, \infty\right)$ *contains a countable dense subset* U *such that for any* $r \in U$ *the inequality* $\pi^3(S_r^2) \geqslant 5$ *holds.*

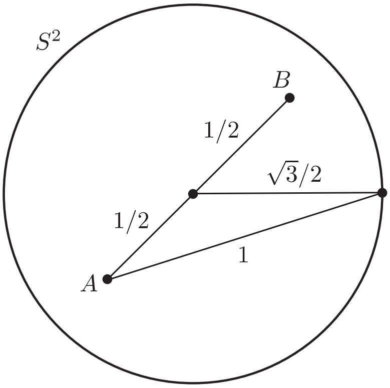

Figure 1

First, we note that Theorem 2 allows us to give a new proof of the known bound $\chi(\mathbb{R}^4) \geqslant 7$. To see this we fix a regular colouring of $\mathbb{R}^4$. Using statement 1) of Theorem 2, we choose in $\mathbb{R}^4$ a two-dimensional sphere S^2 of radius $\frac{\sqrt{3}}{2}$ which has 5 different colours in the given colouring. Draw the line through the centre of the sphere orthogonal to the three-dimensional plane containing this sphere. Then choose on this line two points A and B, each at distance $\frac{1}{2}$ from the centre of the sphere (see Fig. 1). Each of these points is at distance $\sqrt{\frac{3}{4} + \frac{1}{4}} = 1$ from any point of the sphere, and furthermore, $|AB| = 1$. Therefore, the number of colours used to colour the sphere and these two points is at least seven.

Second, we can improve the lower bound for the number $\chi(\mathbb{R}^{12})$. The bound $\chi(\mathbb{R}^9) \geqslant 21$ was proved using a construction on the eight-dimensional sphere S_r^8 of radius $r = \sqrt{\frac{5}{8}}$ (see [19]). Set $r' = \sqrt{1 - \frac{5}{8}} = \sqrt{\frac{3}{8}}$. Fixing a regular colouring of the space $\mathbb{R}^{12}$ and using any of statements 2)–4) of Theorem 2, we can find in $\mathbb{R}^{12}$ a sphere $S_{r'}^2$ which is coloured with at least five colours. We choose a sphere S_r^8 in the nine-dimensional plane orthogonal to the three-dimensional plane containing the sphere $S_{r'}^2$ in such a way that the centres of the two spheres coincide. As a result, we obtain that the number of colours used to colour the union $S_r^8 \cup S_{r'}^2$ is at least $21 + 5 = 26$, since we have chosen r' in such a way that any two points in different spheres are at distance one from each other. Therefore, $\chi(\mathbb{R}^{12}) \geqslant 26$ and this bound is better than the previous bound $\chi(\mathbb{R}^{12}) \geqslant 25$. Now we formulate a theorem which will allow us to improve the lower bound for $\chi(\mathbb{R}^{12})$ even further.

Theorem 3. 1) *If* $n \geqslant 10$ *and* $r > \sqrt{\left(\frac{3}{4}\sqrt{3} - \frac{47}{48}\right)^2 + 2 - \sqrt{3}} \approx 0.6085$, *then* $\pi^n(S_r^2) \geqslant 6$.

2) *If* $n \geqslant 14$ *and* $r > \sqrt{\frac{13}{24}\left(\sqrt{3} - \frac{47}{39}\right)^2 + 2 - \sqrt{3}} \approx 0.6468$, *then* $\pi^n(S_r^2) \geqslant 7$.

Using the same argument as that following Theorem 2, we obtain the following result.

Corollary 1. *We have* $\chi(\mathbb{R}^{12}) \geqslant 27$.

Proof. We only need to check that statement 1) of Theorem 3 is applicable. To do this we note that $\sqrt{\frac{3}{8}} \approx 0.6124$, which implies that both the radius and the dimension satisfy the conditions of the theorem.

3.3. Multidimensional case. In this section we obtain multidimensional analogues of Theorems 1–3. The construction used here will be different from the constructions of the previous sections. Therefore, we shall obtain an alternative proof of the bound $\chi(\mathbb{R}^4) \geqslant 7$ as a byproduct. We shall also show that the inequality $\pi^5(S_r^2) \geqslant 6$ holds for some radius r.

Theorem 4. 1) *The inequality* $\pi^{2n+2}(S_r^n) \geqslant 2n+2$ *holds for* $n \leqslant 7$ *and*

$$r \in \left(\sqrt{\frac{\left(1 + \sqrt{\frac{n}{2(n+1)}}\right)^2}{n^2 + 6n + 4 + \sqrt{8n(n+1)}} + \frac{n}{2n+2}}, \sqrt{\frac{\left(\sqrt{n+2} + \sqrt{2}\right)^2 + n^3}{(2n+2)n^2}}\right).$$

2) *The inequality* $\pi^{2n+1}(S_r^n) \geqslant 2n+2$ *holds for* $n \leqslant 7$ *and*

$$r = \sqrt{\frac{\left(\sqrt{n+2} + \sqrt{2}\right)^2 + n^3}{(2n+2)n^2}}.$$

Remark 1. In the case $n = 2$ statement 2) of Theorem 4 gives that $\pi^5(S_r^2) \geqslant 6$ for $r = \sqrt{\frac{7+2\sqrt{2}}{12}}$.

Theorem 5. *The inequality* $\pi^{n^2+3n+1}(S_r^n) \geqslant 2n+2$ *holds for*

$$r > \sqrt{\frac{(n^4 + 4n^3 + 6n^2 + 4n + 2)^2}{32(n+1)^6((n+1)^2 + 1)} + \frac{n}{2n+2}}.$$

Denote by $\lceil x \rceil$ the upper integral part (ceiling function) of x.

Theorem 6. 1) *The inequality* $\pi^{2n}(S_r^n) \geqslant \lceil \frac{3n+3}{2} \rceil$ *holds for*

$$r \in \left(\sqrt{\frac{\left(1 + \sqrt{\frac{n}{2(n+1)}}\right)^2}{n^2 + 6n + 4 + \sqrt{8n(n+1)}} + \frac{n}{2n+2}}, \sqrt{\frac{\left(\sqrt{n+2} + \sqrt{2n+2}\right)^2 + n^3}{(2n+2)n^2}}\right].$$

2) *The inequality* $\pi^{2n}(S_r^n) \geqslant \lceil \frac{3n+3}{2} \rceil$ *holds for*

$$r \in \left[\sqrt{\frac{\left(\sqrt{2n+2} - \sqrt{n+2}\right)^2 + n^3}{(2n+2)n^2}}, \sqrt{\frac{\left(1 - \sqrt{\frac{n}{2(n+1)}}\right)^2}{n^2 + 6n + 4 - \sqrt{8n(n+1)}} + \frac{n}{2n+2}}\right).$$

Remark 2. Substituting the values $n=2$, $r=\frac{\sqrt{3}}{2}$ into statement 1) of Theorem 6 we obtain $\pi^4(S_r^2)\geqslant\lceil\frac{9}{2}\rceil=5$ (to see this we only need to check that the conditions of the theorem are satisfied for this r). Using the same argument as that following Theorem 2 we obtain a new proof of the bound $\chi(\mathbb{R}^4)\geqslant 7$.

Table 2

n	Theorem 4	Theorem 5	Theorem 6 1)	Theorem 6 2)
1	$r\in(0.63246, 1.65068)$	$r\in(0.52747,\infty)$	$r\in(0.63246, 1.93185)$	$r\in(0.51764, 0.53452)$
2	$r\in(0.65248, 0.90501)$	$r\in(0.60179,\infty)$	$r\in(0.65248, 1.07622)$	$r\in(0.58460, 0.58907)$
3	$r\in(0.66236, 0.74837)$	$r\in(0.63611,\infty)$	$r\in(0.66236, 0.85512)$	$r\in(0.61634, 0.61813)$
4	$r\in(0.66861, 0.70235)$	$r\in(0.65585,\infty)$	$r\in(0.66861, 0.77254)$	$r\in(0.63496, 0.63585)$
5	$r\in(0.67306, 0.68674)$	$r\in(0.66867,\infty)$	$r\in(0.67306, 0.73560)$	$r\in(0.64722, 0.64773)$
6	$r\in(0.67646, 0.68139)$	$r\in(0.67766,\infty)$	$r\in(0.67646, 0.71709)$	$r\in(0.65592, 0.65623)$
7	$r\in(0.67916, 0.67997)$	$r\in(0.68431,\infty)$	$r\in(0.67916, 0.70711)$	$r\in(0.66240, 0.66261)$
8	–	$r\in(0.68944,\infty)$	$r\in(0.68139, 0.70146)$	$r\in(0.66743, 0.66757)$
9	–	$r\in(0.69350,\infty)$	$r\in(0.68325, 0.69817)$	$r\in(0.67143, 0.67154)$
10	–	$r\in(0.69681,\infty)$	$r\in(0.68485, 0.69625)$	$r\in(0.67471, 0.67479)$

Theorem 7. *The inequality $\pi^3(S_r^2)\geqslant 5$ holds for*

$$r\in\biggl\{\sqrt{\frac{3}{4}-\frac{1}{\sqrt{6}}}\approx 0.5846,\ \sqrt{\frac{3}{4}+\frac{1}{\sqrt{6}}}\approx 1.0762\biggr\}.$$

The bounds for the radius from Theorems 4–6 are given in Table 2.

§ 4. Auxiliary statements

We shall need the following lemma from [20].

Lemma 1. *Let $n\geqslant 2$. Fix a regular colouring of $\mathbb{R}^n$. For any $r>0$, there is a pair of points $A,B\in\mathbb{R}^n$ of different colours such that $|AB|=r$.*

We shall also need the following well-known formula for the radius r of the sphere circumscribed around the simplex Sp^n with edge of unit length and $n+1$ vertices (the unit simplex of dimension n):

$$r=\sqrt{\frac{n}{2n+2}}.$$

In what follows a simplex will be understood as the set of its vertices.

The following simple lemma was proved in [20].

Lemma 2. *For any $r>\frac{1}{2}$ and any $\varepsilon>0$ there exists $r_0\in(\frac{1}{2},r)$ such that $|r-r_0|<\varepsilon$ and the circle $S_{r_0}\subset\mathbb{R}^2$ of radius r_0 contains a cycle of odd length each of whose edges has unit length.*

Essentially, this lemma states that the set $\{r\geqslant\frac{1}{2}\}$ contains a countable dense subset of radii r satisfying $\chi(S_r^1)\geqslant 3$. For other radii we can only assert that $\chi(S_r^1)\geqslant 2$. On the other hand, it is easy to show that both these bounds are exact, as long as we accept the axiom of choice. By the Erdős-De Bruijn Theorem (see [1]), if the chromatic number of a metric space U is finite, then it is achieved on a finite subset of U (assuming the axiom of choice). In our situation the finiteness of the chromatic number is clear. We can therefore choose a finite subset Λ on which the number $\chi(S^1_r)$ is achieved, and construct a colouring of Λ. If r belongs to the above described dense subset, then we colour all cycles of odd length with edges of unit length lying in Λ with three colours, and colour other vertices in Λ with any two of these three colours. We observe that there is no pair of vertices of the same colour at distance one from each other, because any vertex is at distance one from only two other vertices. In particular, if this vertex belongs to a cycle of length three, then it is connected by edges of unit length only to vertices from the same cycle. Therefore, three colours are enough. In the case when r does not belong to the dense subset the colouring is constructed similarly, with the only difference that now two colours are enough since there are no cycles of odd length.

For $r < \frac{1}{2}$ we have that $\chi(S^1_r) = 1$, while $\pi^n(S^1_r) \geqslant 2$ for $n \geqslant 2$ by Lemma 1. It follows from statement 3) of Theorem 1 and Lemma 1 that even in the case $n = 2$ the inequality $\chi(S^1_r) < \pi^n(S^1_r)$ holds for almost all radii.

We shall need the following theorem proved by Lovász [23] in 1983.

Theorem 8. *The bound* $\chi(S^n_r) \geqslant n+1$ *holds for* $r > \frac{1}{2}$.

We shall also need the following lemma.

Lemma 3. *Fix a regular colouring of* $\mathbb{R}^{2n}$. *For any* $x > \sqrt{\frac{n}{2n+2}}$, *we can find in* $\mathbb{R}^{2n}$ *a regular* n*-dimensional simplex* Sp^n_x *with edge* x *which is coloured with* $n+1$ *colours. In other words,* $\pi^{2n}(Sp^n_x) = n+1$ *for any* $x > \sqrt{\frac{n}{2n+2}}$.

Proof. We use induction on the dimension of a simplex. For $n = 1$ the lemma is a particular case of Lemma 1. Let $n = k$. By the induction hypothesis, we can find a $(k-1)$-dimensional simplex Sp^{k-1}_x with edge x all of whose k vertices have different colours. Draw a $(k+1)$-dimensional plane α through the centre of Sp^{k-1}_x orthogonal to the plane containing Sp^{k-1}_x. On the plane α we take a k-dimensional sphere S^k all of whose points are at distance x from the points of Sp^{k-1}_x. Using Theorem 8 we can choose a point A on the sphere such that the colour of A is different from the colours of the vertices of Sp^{k-1}_x. Now the point A and Sp^{k-1}_x span the required simplex. It remains to verify the conditions of Theorem 8, that is, that the radius of S^k is bigger than $\frac{1}{2}$. In our case the radius of S^k is equal to the height of the simplex Sp^k_x. Consider the k-dimensional simplex with edge $\sqrt{2}$ in $\mathbb{R}^{k+1}$ whose vertices are $(1,0,\dots,0),\dots,(0,\dots,0,1)$. Its height is equal to the distance $h_{\sqrt{2}}$ between the points $(1,0,\dots,0)$ and $(0,\frac{1}{k},\dots,\frac{1}{k})$:

$$h_{\sqrt{2}} = \sqrt{1 + k\frac{1}{k^2}} = \sqrt{\frac{k+1}{k}}.$$

Then for a simplex with edge x we obtain $h_x = x\sqrt{\frac{k+1}{2k}}$, and $h_x > \frac{1}{2}$ for $x > \sqrt{\frac{k}{2k+2}}$.

Corollary 2. $\pi^{2n}(S^{n-1}_a) \geqslant n+1$ *for any* $a > \frac{n}{2n+2}$.

Proof. It is enough to recall that the radius of the sphere circumscribed around a simplex with edge x is $a = x\sqrt{\frac{n}{2n+2}}$, and then apply Lemma 3.

In a sense, Corollary 2 strengthens Theorem 8, which is used to prove Lemma 3: the bound in Corollary 2 is bigger by one, and the radius of a sphere is bounded from below by the number $\frac{n}{2n+2}$ instead of $\frac{1}{2}$. However, the chromatic number is replaced by the speckledness. In fact, this simply shows that the speckledness is more easily bounded from below.

§ 5. Proofs of the results

5.1. One-dimensional case. Proof of Theorem 1. We first prove statement 2). Assume that $n \geqslant 5$ and $r \in \{\frac{1}{\sqrt{3}}\} \cup [\sqrt{2+\sqrt{3}}, \infty)$.

Consider an arbitrary segment CD of length one and a circle S_r^1 of radius r containing its edges C and D (see Fig. 2). Draw a segment AB with ends on S_r^1 at distance $\sqrt{\frac{2}{5}}$ parallel to CD. Such a segment always exists. Indeed, for $r \geqslant \frac{1}{\sqrt{3}}$ the distance from the centre of CD to the point F (the point of S_r^1 farthest from the centre of CD) is at least $\frac{\sqrt{3}}{2} > \sqrt{\frac{2}{5}}$ (here $\frac{\sqrt{3}}{2}$ is the height of an equilateral triangle with unit edge, whose circumscribed circle has radius $\frac{1}{\sqrt{3}}$).

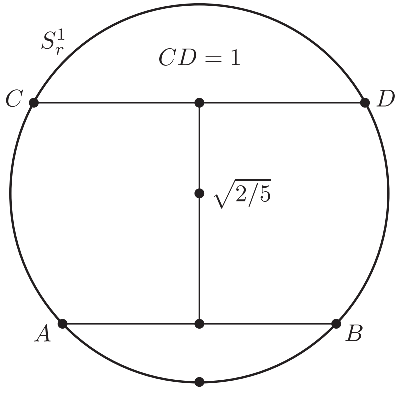

Figure 2

Since the segment CD and the circle S_r^1 have been chosen arbitrarily, we can assume by Lemma 1 that the points A and B have different colours.

Now we shall rotate the space around the line AB. Then each of the points C, D describes a sphere $S_{r'}^{n-2}$ of radius $r' = \sqrt{\frac{2}{5}}$. We denote the first sphere by S_C and the second by S_D. Note that our r' is exactly the radius of the circumscribed sphere for the simplex Sp^4. Hence, for $n \geqslant 5$, we can choose points $U_1, \dots, U_5$ on S_C at distance one from each other. Denote by ω_i, $i = 1, \dots, 5$, the rotation of the space taking C to U_i. Set $V_i = \omega_i(D) \in S_D$, $i = 1, \dots, 5$. There are at least three points among the U_i whose colours are different from those of A and B. The same is true for the points V_i. By the Dirichlet principle, there is j such that neither of

U_j and V_j has the same colour with A or B. The points U_j and V_j cannot have the same colour since they are at distance one from each other. We therefore obtain points A, B, U_j, V_j, all of different colours. Furthermore, they all belong to the circle S^1_r. Thus, $\pi^n(S^1_r) \geqslant 4$, which proves statement 2).

Next we prove statement 3). Now let $n \geqslant 2$ and $r > \frac{1}{2}$. The proof is similar to the first part of the proof of statement 2). We choose an arbitrary point C and a circle S^1_r containing it. Then choose a segment AB in such a way that $A \in S^1_r$, $B \in S^1_r$ and $|AC| = |BC| = 1$. This can be done since $r > \frac{1}{2}$. Moreover, since C is chosen arbitrarily, we may assume that the end-points of AB have different colours (once again using Lemma 1). We therefore obtain a three-coloured triangle ABC, which proves statement 3).

Now we prove statement 4). Let $n \geqslant 3$ and $r > \frac{1}{4}$. Choose a circle S^1_r arbitrarily and inscribe into it an isosceles triangle of height h from the vertex C to the base AB. The number h is chosen in such a way that it is equal to the radius of a circle which contains an inscribed cycle of odd length with unit edges. This can be achieved by varying the length of the segment AB and using Lemma 2. The latter lemma is applicable since $|h|$ can assume any value in the interval $(0, 2r)$, and the diameter of S^1_r is equal to $2r$, which is bigger than $\frac{1}{2}$. By applying Lemma 1 we may assume that the end-points of the segment AB have different colours. Now we rotate the space $\mathbb{R}^3$ around AB. The point C describes a circle which has an inscribed cycle of odd length, which implies that this circle is coloured with at least three colours. The circle contains a point D whose colour is different from the colours of A and B. The points A, B, D form a three-coloured triangle inscribed into a circle of radius r. Thus, statement 4) is proved.

Finally, we prove statement 5). Let $n = 3$. We only sketch the proof, as it follows closely the proof of statement 1) given in [20]. We inscribe into S^1_r points A, P, Q, B successively in such a way that $|AP| = |PQ| = |QB| = 1$. By applying Lemma 1 we may assume without loss of generality that the points A and B have different colours. Then, by Lemma 2, there exists a countable dense subset $U \subset (\frac{1}{2}, \infty)$ consisting of such r' that a circle $S^1_{r'}$ of radius $r' \in U$ has an inscribed cycle of odd length with unit edges. By repeating the calculations from [20] it is easy to see that for

$$r \in \left(\sqrt{\frac{1}{2+\sqrt{3}}}, \frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \sqrt{2+\sqrt{3}}\right)$$

the distance r' from P to the line AB is bigger than $\frac{1}{2}$. Therefore, we can find a countable dense set U,

$$U \subset \left(\sqrt{\frac{1}{2+\sqrt{3}}}, \frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \sqrt{2+\sqrt{3}}\right)$$

such that $S^1_{r'}$ has an inscribed cycle of odd length whenever $r \in U$.

Fix $r \in U$ and rotate the space around the line AB. The point P describes a circle $S^1_{r'}$. We inscribe a cycle of odd length into it and choose an edge P_1P_2 whose colour is different from that of B. The two end-points of this edge are the images of P under some rotations of the space. By applying these rotations to Q we obtain two vertices Q_1, Q_2. The colour of one of them (say, Q_1) is different from the colour of A. Then the vertices A, P_1, Q_1, B have all different colours.

5.2. Two-dimensional case.

5.2.1. *Proof of Theorem* 2. We first prove statement 1). By the assumption, we have $r > \frac{1}{2\sqrt{\sqrt{3}-1}}$, $r \neq \sqrt{\frac{3}{8}}$ and $n \geqslant 4$.

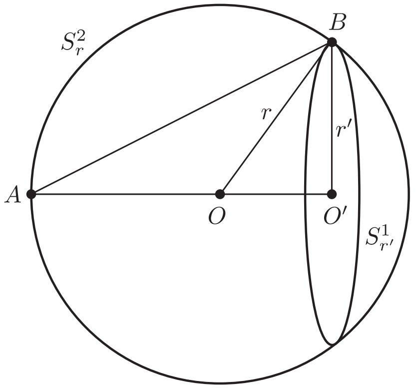

Figure 3

Take any point A on the sphere S_r^2 (see Fig. 3). Let $S_{r'}^1 \subset S_r^2$ be a circle consisting of points at distance one from A. Then we have the following relations for r and r':

$$|OO'| = \sqrt{r^2 - r'^2},$$

$$\begin{aligned} 1 = |AB|^2 = r'^2 + (r + |OO'|)^2 = r'^2 + \left(r + \sqrt{r^2 - r'^2}\,\right)^2 = 2r^2 + 2r\sqrt{r^2 - r'^2} \\ \implies \quad 4r^2(r^2 - r'^2) = 1 - 4r^2 + 4r^4 \quad \implies \quad 4r^2(1 - r'^2) = 1 \\ \implies \quad r = \frac{1}{2\sqrt{1 - r'^2}}. \end{aligned}$$

Clearly, the inequality $\chi'(S_{r'}^1) \geqslant k$ implies that $\chi'(S_r^2) \geqslant k + 1$. Take

$$r' \in \left(\sqrt{\frac{1}{2 + \sqrt{3}}}, \frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, 1\right).$$

Since the point A and the sphere S_r^2 have been chosen arbitrarily, we can use statement 1) of Theorem 1 to choose $S_{r'}^1$ in such a way that $\chi'(S_{r'}^1) \geqslant 4$. Note that this is the place where we use the assumption $n \geqslant 4$. The restrictions on r' imply the following restrictions on r:

$$r > \frac{1}{2\sqrt{1 - \frac{1}{2+\sqrt{3}}}} = \frac{1}{2\sqrt{1 - (2 - \sqrt{3})}} = \frac{1}{2\sqrt{\sqrt{3} - 1}}, \qquad r \neq \frac{1}{2\sqrt{1 - \frac{1}{3}}} = \sqrt{\frac{3}{8}}.$$

The restriction $r' < 1$ does not impose any restrictions on r, since $r \to \infty$ as $r' \to 1$. Statement 1) is therefore proved.

Now we prove statement 2). The proof is similar, with the only difference that we use statement 2) of Theorem 1 instead of statement 1). This is why the method works starting from the dimension five instead of four.

The proof of statement 5) is also analogous to that of statement 1), but this time we use statement 5) of Theorem 1.

Now let us prove statement 3). Let $r > \frac{1}{\sqrt{10}} + \sqrt{\frac{5}{8}}(2-\sqrt{3})$ and $n \geqslant 6$.

Once again we shall need a four-coloured circle $S^1_{r'}$, obtained using statements 1) or 2) of Theorem 1, and a point A, however this time we require that $|AO'| = \sqrt{\frac{2}{5}}$. If $\chi'(S^1_{r'}) \geqslant 5$ then there is nothing to prove. Otherwise we rotate the space around the plane containing the circle $S^1_{r'}$. The point A describes a sphere $S^{n-3}_{r_1}$ where $r_1 = \sqrt{\frac{2}{5}}$ is the radius of a three-dimensional sphere circumscribed around a four-dimensional simplex Sp^4 with unit edges. By the assumption, $n-3 \geqslant 3$. Hence, $S^{n-3}_{r_1} \supset S^3_{r_1} \supset Sp^4$. Take an arbitrary simplex $Sp^4 \subset S^3_{r_1}$. The colour of one of its vertices is different from the colours of the circle $S^1_{r'}$. Since this vertex and the circle $S^1_{r'}$ lie on the sphere S^2_r, the latter sphere is coloured with at least five colours. It remains to determine for which values of the radius r the value $|AO'|$ can be equal to $\sqrt{\frac{2}{5}}$ (see Fig. 3). We have

$$|AO'| = \sqrt{\frac{2}{5}} = r + \sqrt{r^2 - r'^2} \quad \Longrightarrow \quad \frac{2}{5} + r^2 - \sqrt{\frac{8}{5}}r = r^2 - r'^2$$
$$\Longrightarrow \quad r = \frac{1}{\sqrt{10}} + \sqrt{\frac{5}{8}}r'^2.$$

Since here we can apply both statements 1) and 2) of Theorem 1 (as $n \geqslant 6$), the only restriction on r' is

$$r' > \frac{1}{\sqrt{2+\sqrt{3}}} = \sqrt{2-\sqrt{3}}.$$

This implies the following restriction on r:

$$r > \frac{1}{\sqrt{10}} + \sqrt{\frac{5}{8}}(2-\sqrt{3}).$$

Finally, we prove statement 4). We have $r > \frac{1}{\sqrt{2+\sqrt{3}}}$ and $n \geqslant 7$.

Choose a circle $S^1_{r'}$ of radius $r' = \frac{1}{\sqrt{2+\sqrt{3}}} + \varepsilon \leqslant r$ and a sphere $S^2_r \supset S^1_{r'}$ in such a way that the circle $S^1_{r'}$ is coloured with four colours. This can be done by statements 1) and 2) of Theorem 1. If $\chi'(S^1_{r'}) \geqslant 5$ then there is nothing to prove. Otherwise the distance between the point A and the circle $S^1_{r'}$ is bigger than $\frac{1}{2}$ (see Fig. 3):

$$|AO'| = r + \sqrt{r^2 - r'^2} > r > r' = \frac{1}{\sqrt{2+\sqrt{3}}} + \varepsilon > \frac{1}{2}.$$

Now we rotate the space around the plane containing the circle $S^1_{r'}$. The point A describes a sphere $S^{n-3}_{r_1}$, that is an $(n-3)$-dimensional sphere of radius

$r_1 = |AO'| > \frac{1}{2}$, and we can apply Theorem 8. By the assumption, $n-3 \geqslant 4$. Hence, the sphere $S^{n-3}_{r_1}$ is coloured with at least five colours, and we may choose on it a point A' whose colour is different from the colours of the circle $S^1_{r'}$. The point A' and the circle $S^1_{r'}$ lie on the sphere S^2_r. The theorem is proved.

5.2.2. *Proof of Theorem* 3. The proof of both statements of the theorem is very similar to the proof of statement 3) of Theorem 2, with the only difference that we replace the point A from Theorem 2 by a segment of unit length in the case of statement 1) and by an equilateral triangle in the case of statement 2) (see Fig. 4). Then we use an argument similar to that from the proof of statement 2) of Theorem 1.

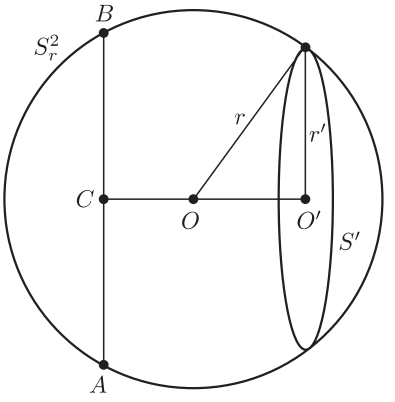

Figure 4

We first prove statement 1). The conditions on the radius and the dimension are as follows:

$$r > \sqrt{\left(\frac{3}{4}\sqrt{3} - \frac{47}{48}\right)^2 + 2 - \sqrt{3}}, \qquad n \geqslant 10.$$

By an application of statements 1), 2) of Theorem 1 we can choose a circle S' of radius $r' > \sqrt{2-\sqrt{3}}$ which is coloured with at least four colours. Then we draw a segment AB of unit length in such a way that its centre C is at the same distance from all points of the circle S', and the segment is parallel to the plane containing the circle. We also require that $|CO'| = \frac{2}{3}$, where O' is the centre of the circle S'. It is easy to see that the points A, B and the circle S' lie on a two-dimensional sphere S^2_r with centre on the line CO'. Then we rotate the points A, B around the plane containing S'. Under this rotation each of the points A, B describes an $(n-3)$-dimensional sphere S^{n-3}, where $n-3 \geqslant 7$ by the assumption. The radius of the sphere, equal to $\frac{2}{3}$, and the dimension are chosen so that the sphere S^{n-3} contains Sp^8, a unit simplex with nine vertices. We consider the two copies Sp_A and Sp_B of this simplex, obtained by the rotation of the vertices A and B. We can choose a pair of vertices $A' \in Sp_A$, $B' \in Sp_B$ in such a way that neither of the colours of A' and B' coincides with any of the four colours of the circle (it is clear that if the circle is coloured with five colours, then we may use the technique of Theorem 2, and the bounds for both the radius and the dimension will be improved). Since the colours of A' and B' are different, we obtain that at least six colours are required for the colouring of S' together with A' and B', and therefore for the colouring of the whole S_r^2. Now we estimate the radius of S_r^2 for which this technique can be implemented, using the restriction $r' > \sqrt{2-\sqrt{3}}$ arising from the application of Theorem 1.

Case 1. The centre O of the sphere S_r^2 does not belong to the segment CO'. Then it is clear that the point O' is closer to O than the point C. Hence,

$$r = \sqrt{|CO|^2 + \frac{1}{4}} \geqslant \sqrt{|CO'|^2 + \frac{1}{4}} = \sqrt{\frac{4}{9} + \frac{1}{4}} = \sqrt{\frac{25}{36}} = \frac{5}{6}.$$

The relations for r and r' are as follows:

$$\frac{2}{3} = |CO'| = |CO| - |OO'| = \sqrt{r^2 - \frac{1}{4}} - \sqrt{r^2 - r'^2}$$
$$\implies \quad r^2 - \frac{1}{4} = r^2 - r'^2 + \frac{4}{9} + \frac{4}{3}\sqrt{r^2 - r'^2} \quad \implies \quad \frac{4}{3}\sqrt{r^2 - r'^2} = r'^2 - \frac{25}{36}.$$

The latter equation has a solution provided that $r \geqslant \sqrt{\frac{25}{36}} = \frac{5}{6}$, and the solution r' satisfies $r' \geqslant \sqrt{\frac{25}{36}} = \frac{5}{6} > \sqrt{2-\sqrt{3}}$. Hence, in this case no additional restrictions on r appear.

Case 2. The point O belongs to the segment CO'. Then

$$r = \sqrt{|CO|^2 + \frac{1}{4}} \leqslant \sqrt{|CO'|^2 + \frac{1}{4}} = \sqrt{\frac{4}{9} + \frac{1}{4}} = \sqrt{\frac{25}{36}} = \frac{5}{6}.$$

The relations for r and r' are as follows:

$$\frac{2}{3} = |CO'| = |CO| + |OO'| = \sqrt{r^2 - \frac{1}{4}} + \sqrt{r^2 - r'^2}$$
$$\implies \quad r^2 - \frac{1}{4} = r^2 - r'^2 + \frac{4}{9} - \frac{4}{3}\sqrt{r^2 - r'^2} \quad \implies \quad \frac{4}{3}\sqrt{r^2 - r'^2} = \frac{25}{36} - r'^2$$
$$\implies \quad r^2 - r'^2 = \left(\frac{25}{48} - \frac{3}{4}r'^2\right)^2 \quad \implies \quad r = \sqrt{\left(\frac{25}{48} - \frac{3}{4}r'^2\right)^2 + r'^2}.$$

It is easy to see that the root on the right hand side of the latter identity grows with the growth of r', and therefore the restrictions on r are exactly those listed in the theorem, namely,

$$r > \sqrt{\left(\frac{25}{48} - \frac{3}{4}(2-\sqrt{3})\right)^2 + 2 - \sqrt{3}} = \sqrt{\left(\frac{3}{4}\sqrt{3} - \frac{47}{48}\right)^2 + 2 - \sqrt{3}}.$$

Now we prove statement 2). By the assumption, $r > \sqrt{\frac{13}{24}(\sqrt{3} - \frac{47}{39})^2 + 2 - \sqrt{3}}$ and $n \geqslant 14$.

This statement is proved in a similar way. The segment AB is replaced by an equilateral triangle with unit edge on the plane parallel to the plane containing S'. The centre of the segment is replaced by the centre of the circumscribed circle for the triangle, and the value $\frac{2}{3}$ is replaced by $\sqrt{\frac{6}{13}}$. Since the appropriate colours need to be chosen for three vertices instead of two, we use Sp^{12} instead of Sp^8. Therefore, the dimension from which the method works increases by four. By repeating the calculation from Case 1 of the proof of statement 1) we show that there are still no further restrictions on r. We therefore can repeat the calculation from Case 2 with $|AC| = \frac{1}{2}$ replaced by the radius of the circumscribed circle, equal to $\frac{1}{\sqrt{3}}$, and $|CO'| = \frac{2}{3}$ replaced by $|CO'| = \sqrt{\frac{6}{13}}$:

$$\begin{aligned}
\sqrt{\frac{6}{13}} &= |CO'| = |CO| + |OO'| = \sqrt{r^2 - \frac{1}{3}} + \sqrt{r^2 - r'^2} \\
&\implies \quad r^2 - \frac{1}{3} = r^2 - r'^2 + \frac{6}{13} - 2\sqrt{\frac{6}{13}}\sqrt{r^2 - r'^2} \\
&\implies \quad 2\sqrt{\frac{6}{13}}\sqrt{r^2 - r'^2} = \frac{31}{39} - r'^2 \quad \implies \quad r^2 - r'^2 = \frac{13}{24}\left(\frac{31}{39} - r'^2\right)^2 \\
&\implies \quad r = \sqrt{\frac{13}{24}\left(\frac{31}{39} - r'^2\right)^2 + r'^2}.
\end{aligned}$$

The root on the right hand side of the latter identity grows with the growth of r', and therefore we obtain the required bound, namely,

$$r > \sqrt{\frac{13}{24}\left(\frac{31}{39} - 2 + \sqrt{3}\right)^2 + 2 - \sqrt{3}} = \sqrt{\frac{13}{24}\left(\sqrt{3} - \frac{47}{39}\right)^2 + 2 - \sqrt{3}}.$$

The theorem is proved.

5.3. Multidimensional case.

5.3.1. *Main construction.* Before we go ahead with the proof of the theorems, we analyse a construction to be used later.

Our aim is to find a sphere S_r^n in a certain dimension k, the smaller the better, which is coloured with the maximal possible number of colours. The construction we consider here is as follows (the case $n = 2$ is shown in Fig. 5).

1) We consider a sphere S^{n-1} of a varied radius, and inscribe into it a simplex Sp^n coloured with the maximal possible number of colours. Usually the number of colours will be $n + 1$.
2) On a sphere S_r^n containing S^{n-1} we choose $n + 1$ points, each at distance one from some n vertices of the simplex Sp^n and at distance one from the other n of these points. These $n + 1$ points form a simplex Sp_1^n with edge one inscribed in a sphere $S_{r'}^{n-1}$, $r' = \sqrt{\frac{n}{2n+2}}$, which is embedded in S_r^n and lies on the n-dimensional plane parallel to the similar plane for the sphere S^{n-1}. Once again we point out that each vertex X of the simplex Sp_1^n is at distance one from exactly n vertices of the simplex Sp^n and is not at distance one from exactly one of its vertices. Denote this vertex by A_X.

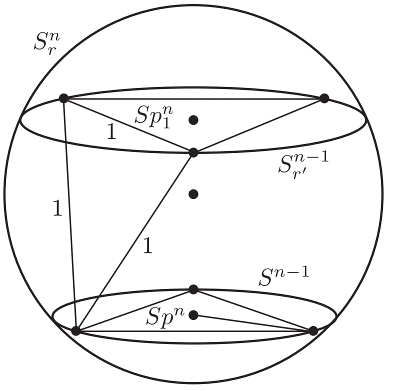

Figure 5

3) We rotate the simplex Sp_1^n around the plane containing S^{n-1}. We choose such a rotation ω for which the points X_i and A_{X_i} have different colours for the maximal number of vertices $X_i \in \omega(Sp_1^n)$.

We comment upon these three items. To implement the first one we shall use Lemma 3, which works for all dimensions except three, and we shall use Lemma 1 for the dimension three. We note that the simplex can be replaced by any construction giving a sphere S^{n-1} coloured with the maximal possible number of colours. However, then we shall not be able to proceed according to item 2), and a much bigger dimension of the ambient space will be required in order to exclude the colours of the sphere fixed under the rotation from the colours of the simplex.

Now we comment on the second item. First we discuss the relative position of the simplices Sp_1^n and Sp^n. It is clear that the line connecting the centres of the spheres S^{n-1} and $S_{r'}^{n-1}$ is perpendicular to the planes containing these spheres, and therefore it contains the centre of S_r^n. This follows from the fact that the distance from the centre of one sphere to any other point on the other sphere is the same by condition 2). We project one sphere orthogonally onto the plane containing the other sphere. Then their centres will coincide (the two-dimensional case is shown in Fig. 6). The projection X' of the vertex $X \in Sp_1^n$ must belong to the line containing one of the heights h of the simplex Sp^n, as otherwise it cannot be at the same distance from the n vertices of the face $\gamma \subset Sp^n$ perpendicular to h. The same can be said about any other vertex of Sp^n. Then the only one indeterminacy remains. Assume that the radius r'' of the sphere S^{n-1} is at most r' (the other case is considered similarly). Then there are two possible locations of the point X' with respect to the plane $\pi \supset \gamma$: it can be on the same side as the vertex $A_X \in Sp^n$, or on the other side. We shall consider both possibilities.

We note also that if we do not require that the distance from X to the vertices of γ is one, then the dimension of the ambient space would increase.

As for the third item, we note that, under the rotation, every vertex of Sp_1^n describes a $(k-n-1)$-dimensional sphere. Depending on the radius of this sphere, we shall inscribe in it a regular simplex of an appropriate dimension with unit edge.

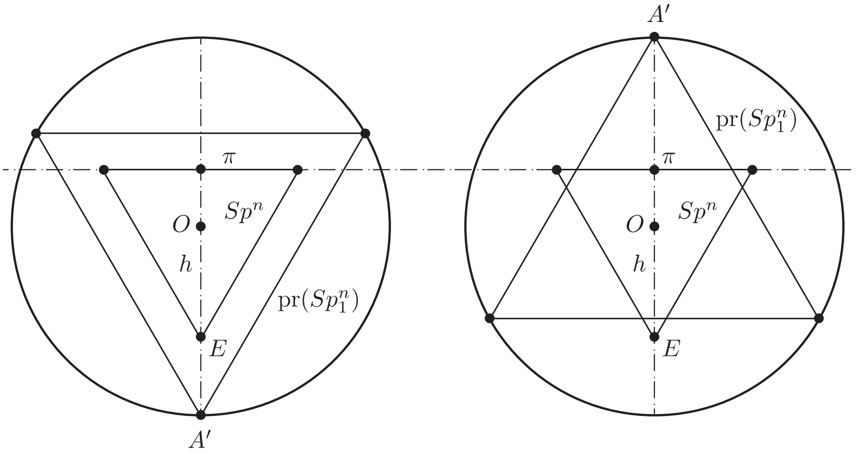

Figure 6

Now we calculate the distance l between the centres of S^{n-1} and $S^{n-1}_{r'}$ and the radius r of the sphere S^n_r in terms of the radius r'' of the sphere S^{n-1} and the dimension n. We first consider case 1 shown in Fig. 7.

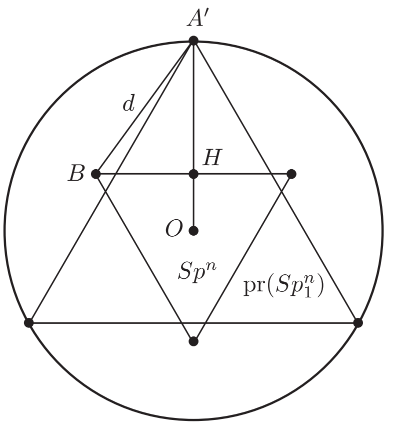

Figure 7

Case 1. We have

$$l=\sqrt{1-d^2}, \qquad d^2=|A'H|^2+|BH|^2,$$

where H is the centre of the sphere circumscribed around the $(n-1)$-dimensional face of Sp^n whose vertices are at the same distance from A'. Therefore,

$$|BH|^2=\left(r''^2\,\frac{2n+2}{n}\right)\frac{n-1}{2n}=r''^2\,\frac{n^2-1}{n^2},$$

where the factor in the parentheses is the square of the edge length of the simplex Sp^n,

$$|A'H|^2 = (|A'O| - |OH|)^2, \qquad |A'O| = \sqrt{\frac{n}{2n+2}}.$$

Clearly, $|OH|$ is the difference between the height of the simplex with edge $r''\sqrt{\frac{2n+2}{n}}$, which was calculated in Lemma 3, and the radius of its circumsphere. We have

$$|OH| = r''\sqrt{\frac{2n+2}{n}}\left(\sqrt{\frac{n+1}{2n}} - \sqrt{\frac{n}{2n+2}}\right) = r''\sqrt{\frac{2n+2}{n}}\left(\frac{n+1-n}{\sqrt{n(2n+2)}}\right) = \frac{r''}{n}.$$

Substituting the obtained expressions into the formula for l, we obtain

$$l^2 = 1 - r''^2\frac{n^2-1}{n^2} - \left(\sqrt{\frac{n}{2n+2}} - \frac{r''}{n}\right)^2 = 1 - r''^2 - \frac{n}{2n+2} + r''\sqrt{\frac{2}{n(n+1)}}. \quad (5.1)$$

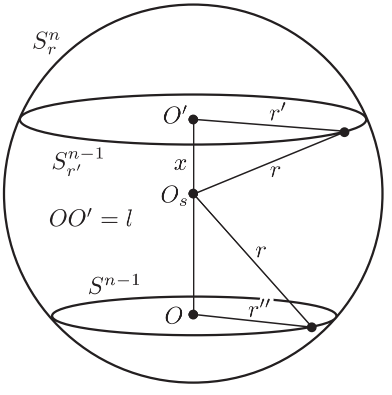

Figure 8

Now we shall find an expression for the radius r of the sphere S^n_r using formula (5.1). Here O, O' and O_s are the centres of S^{n-1}, $S^{n-1}_{r'}$ and S^n_r respectively, and r'', r', r are the radii of these spheres (see Fig. 8):

$$r^2 = x^2 + r'^2 = (l-x)^2 + r''^2 \quad\Longrightarrow\quad x^2 + \frac{n}{2n+2} = (l-x)^2 + r''^2$$

$$\Longrightarrow\quad 2lx = l^2 + r''^2 - \frac{n}{2n+2} \overset{(5.1)}{=} 1 - \frac{n}{n+1} + r''\sqrt{\frac{2}{n(n+1)}}$$

$$= \frac{1}{n+1} + r''\sqrt{\frac{2}{n(n+1)}}. \quad (5.2)$$

Substituting (5.2) into the formula for the radius, we obtain

$$r^2 = \frac{\left(\frac{1}{n+1} + r''\sqrt{\frac{2}{n(n+1)}}\right)^2}{4l^2} + \frac{n}{2n+2}. \quad (5.3)$$

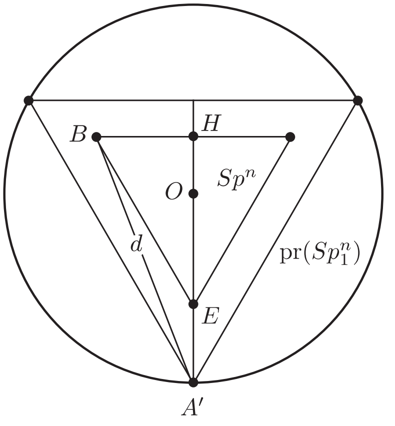

Figure 9

Case 2. Now we shall carry out similar calculations in the case shown in Fig. 9. We have

$$l=\sqrt{1-d^2}, \qquad d^2=|A'H|^2+|BH|^2,$$

where H is the centre of the sphere circumscribed around the $(n-1)$-dimensional face of Sp^n whose vertices are at the same distance from A'. Therefore, the length $|BH|$ can be calculated using the same formula as that from Case 1,

$$|A'H|^2=(|A'O|+|OH|)^2, \qquad |A'O|=\sqrt{\frac{n}{2n+2}}, \qquad |OH|=\frac{r''\sqrt{\frac{2n+2}{n}}}{\sqrt{n(2n+2)}}=\frac{r''}{n},$$

since $|OH|$ is calculated in the same way as in Case 1. Substituting the obtained expression into the formula for l, we obtain

$$l^2=1-r''^2\frac{n^2-1}{n^2}-\biggl(\sqrt{\frac{n}{2n+2}}+\frac{r''}{n}\biggr)^2=1-r''^2-\frac{n}{2n+2}-r''\sqrt{\frac{2}{n(n+1)}}. \tag{5.4}$$

Now, using formula (5.4) and the notation of Case 1 (see Fig. 8), we can express the radius r of the sphere S_r^n. We obtain a similar formula, namely,

$$2lx=l^2+r''^2-\frac{n}{2n+2}\overset{(5.4)}{=}1-\frac{n}{n+1}-r''\sqrt{\frac{2}{n(n+1)}}=\frac{1}{n+1}-r''\sqrt{\frac{2}{n(n+1)}}. \tag{5.5}$$

Finally, substituting (5.5) into the formula for the radius, we obtain

$$r^2=\frac{\Bigl(\frac{1}{n+1}-r''\sqrt{\frac{2}{n(n+1)}}\Bigr)^2}{4l^2}+\frac{n}{2n+2}. \tag{5.6}$$

5.3.2. *Proof of Theorem 4*. The proof follows the scheme described in the previous subsection. Both statements in the theorem correspond to Case 1 considered above. In the first statement the dimension k of the ambient space is equal to $2n+2$, and in the second statement it is $2n+1$.

We place on an n-dimensional plane α a simplex Sp^n with the radius of the circumsphere equal to a. Using Lemma 3 we obtain $\pi^{2n}(Sp^n)=n+1$ for $a>\frac{n}{2n+2}$. The further argument follows the scheme from the previous subsection. Under the rotation of the space around the plane α each vertex of Sp_1^n describes an $(n+1)$-dimensional sphere in the case of statement 1) or an n-dimensional in the case of statement 2).

Now take an arbitrary vertex $X\in Sp_1^n$ and choose spatial rotations $\omega_1,\dots,\omega_{n+2}$ such that $|\omega_i(X)\omega_j(X)|=1$, $i,j=1,\dots,n+2$, $i\neq j$. In other words, the vertices $\omega_i(X)$ form an $(n+1)$-dimensional simplex Sp_1^{n+1} with unit edge. In order for this to be possible, in the case of statement 1) the radius of the sphere must not be less than the radius of the sphere circumscribed around the simplex, and in the case of statement 2) the two radii must be equal. Then we apply ω_i to all other vertices of Sp_1^n. We obtain $n+1$ copies of the simplex Sp_1^{n+1}. Look at the colours in the colouring of Sp^n. Among these colours, only the colour of A_X may appear in the colouring of $\omega_i(X)$, since any of the vertices from the latter set is at distance one from all vertices of Sp^n except A_X. Then among the $n+2$ vertices $\omega_i(X)$ there are at least $n+1$ whose colours are different from that of A_X. The same is true for the other n vertices of Sp_1^n. We therefore have $n+1$ subsets of an $(n+2)$-element set, each of cardinality $n+1$. Their intersection is nonempty by the Dirichlet principle.

It follows that there is at least one rotation of the space, say ω_1, for which the colours of all points $\omega_1(X_1),\dots,\omega_1(X_{n+1})$ (here X_i are the vertices of Sp_1^n) are different from the colours of any of $A_{X_1},\dots,A_{X_{n+1}}\in Sp^n$ and are different from each other. We therefore obtain two n-dimensional simplices on the n-dimensional sphere S_r^n whose colouring uses $2n+2$ colours. Hence, the bound $\pi^{2n+j}(S_r^n)\geqslant 2n+2$ holds, where we have $j=1$ or $j=2$ depending on the statement being proved. Denote by α' the plane containing $\omega_1(Sp_1^n)$.

It remains to determine the conditions on a guaranteeing that the distance l between the planes α and α' is not less than the radius of the sphere circumscribed around Sp_1^{n+1}. The distance l has been calculated in the previous subsection (formula (5.1)). Having derived the restrictions on a and using the fact that $a>\frac{n}{2n+2}$, we obtain from (5.3) the restrictions on the radius of the sphere. Now we consider both statements of the theorem. We need to verify the condition $l^2\geqslant\frac{n+1}{2n+4}$. In our case it reduces to the solution of the following quadratic inequality:

$$
\begin{aligned}
&\frac{n+1}{2n+4}\leqslant 1-a^2-\frac{n}{2n+2}+a\sqrt{\frac{2}{n(n+1)}}\\
&\qquad\Longrightarrow\quad a^2-a\sqrt{\frac{2}{n(n+1)}}-\frac{2n+3}{2(n+1)(n+2)}\leqslant 0\\
&\qquad\Longrightarrow\quad \sqrt{\frac{1}{2n(n+1)}}-\sqrt{\frac{n+1}{n(n+2)}}\leqslant a\leqslant\sqrt{\frac{1}{2n(n+1)}}+\sqrt{\frac{n+1}{n(n+2)}}.
\end{aligned}
$$

The left inequality is evidently satisfied since $a > \frac{n}{2n+2}$. This implies the following restrictions on a:

$$\frac{n}{2n+2} < a \leqslant \sqrt{\frac{1}{2n(n+1)}} + \sqrt{\frac{n+1}{n(n+2)}}.$$

Clearly, the right-hand side decreases and the left-hand side increases as the dimension grows. It is easy to check (by substitution) that the double inequality has a solution for $n = 7$, and does not have a solution for $n = 8$. This implies the restriction on n in the theorem. Now we look at how r depends on a. We shall show that under the given restrictions the value r^2 grows with the growth of a. To do this we write the expression for r^2 (formula (5.3)) in the following form:

$$r^2 = \frac{1}{4}\,\frac{f^2}{f+g} + \frac{n}{2n+2} = \frac{1}{4}f\biggl(1 - \frac{g}{f+g}\biggr) + \frac{n}{2n+2},$$

where

$$f = \frac{1}{n+1} + a\sqrt{\frac{2}{n(n+1)}}, \qquad g = l^2 - f = \frac{n}{2n+2} - a^2.$$

It is easy to see that f increases monotonically (in a), while g decreases monotonically. In this case both $1 - \frac{g}{f+g}$ and r^2 are monotonically increasing functions. Hence, the value of r is maximised when a is minimised, and vice versa. We therefore obtain the following restrictions on r:

$$r^2 - \frac{n}{2n+2} = \frac{\Bigl(\frac{1}{n+1} + a\sqrt{\frac{2}{n(n+1)}}\Bigr)^2}{4l^2} > \frac{\Bigl(\frac{1}{n+1} + \frac{n}{2n+2}\sqrt{\frac{2}{n(n+1)}}\Bigr)^2}{4\Bigl(1 - \bigl(\frac{n}{2n+2}\bigr)^2 - \frac{n}{2n+2} + \frac{n}{2n+2}\sqrt{\frac{2}{n(n+1)}}\Bigr)}$$

$$\implies \quad r^2 > \frac{\Bigl(1 + \sqrt{\frac{n}{2(n+1)}}\Bigr)^2}{n^2 + 6n + 4 + \sqrt{8n(n+1)}} + \frac{n}{2n+2}.$$

In order to estimate the radius r from above we note that $l^2 = \frac{n+1}{2n+4}$. We make a similar calculation, namely,

$$\begin{aligned} r^2 - \frac{n}{2n+2} &= \frac{\Bigl(\frac{1}{n+1} + a\sqrt{\frac{2}{n(n+1)}}\Bigr)^2}{4l^2} \\ &\leqslant \frac{\Bigl(\frac{1}{n+1} + \Bigl(\sqrt{\frac{1}{2n(n+1)}} + \sqrt{\frac{n+1}{n(n+2)}}\Bigr)\sqrt{\frac{2}{n(n+1)}}\Bigr)^2}{4\bigl(\frac{n+1}{2n+4}\bigr)} \end{aligned}$$

$$\implies \quad r^2 \leqslant \frac{\Bigl(\frac{1}{n} + \frac{1}{n}\sqrt{\frac{2}{n+2}}\Bigr)^2}{2\bigl(\frac{n+1}{n+2}\bigr)} + \frac{n}{2n+2} = \frac{(\sqrt{n+2} + \sqrt{2}\ bigr)^2 + n^3}{(2n+2)n^2}.$$

It remains to note that in the case of statement 2) we have $l = \frac{n+1}{2n+4}$ and the latter inequality turns into an identity. Taking the square root of the inequality above we obtain the required bounds from statements 1) and 2).

We could have tried to prove a similar theorem using the construction from Case 2 considered in the previous subsection. Formula (5.4) implies similar restrictions on a, which arise from the inequality $l^2 \geqslant \frac{n+1}{2n+4}$, namely,

$$\frac{n+1}{2n+4} \leqslant 1 - a^2 - \frac{n}{2n+2} - a\sqrt{\frac{2}{n(n+1)}}$$
$$\Longrightarrow \quad a^2 + a\sqrt{\frac{2}{n(n+1)}} - \frac{2n+3}{2(n+1)(n+2)} \leqslant 0$$
$$\Longrightarrow \quad -\sqrt{\frac{1}{2n(n+1)}} - \sqrt{\frac{n+1}{n(n+2)}} \leqslant a \leqslant \sqrt{\frac{n+1}{n(n+2)}} - \sqrt{\frac{1}{2n(n+1)}}.$$

The left inequality is obviously satisfied. This implies the following restrictions on a:

$$\frac{n}{2n+2} < a \leqslant \sqrt{\frac{n+1}{n(n+2)}} - \sqrt{\frac{1}{2n(n+1)}}.$$

It is easy to check that the right-hand side decreases with the growth of n (for natural n), the double inequality has a solution for $n = 1$, and does not have a solution for $n = 2$. Therefore, the construction of Case 2 can only be used to give circles of bigger speckledness.

5.3.3. *Proof of Theorem* 5. In this subsection we leave out the condition that each vertex of the simplex Sp_1^n is at distance one from n vertices of the simplex Sp^n.

The construction used here is almost the same as that used in the proof of Theorem 4, but this time the simplex Sp_1^n is chosen in the plane α' arbitrarily. However, we require the distance l between the planes α and α' containing Sp^n and Sp_1^n respectively to be equal to the radius of the sphere circumscribed around an $(n+1)^2$-dimensional simplex, that is,

$$l = \frac{n+1}{\sqrt{2(n+1)^2+2}}.$$

This will be achieved by the appropriate choice of a. The further argument follows closely the similar argument in the proof of Theorem 4.

We rotate the space around the plane α. Under this rotation, each vertex of Sp_1^n describes an (n^2+2n)-dimensional sphere of radius l. Take an arbitrary vertex $X \in Sp_1^n$ and choose rotations $\omega_1, \dots, \omega_{n^2+2n+2}$ of the space such that

$$|\omega_i(X)\omega_j(X)| = 1, \qquad i, j = 1, \dots, n^2+2n+2, \quad i \neq j.$$

In other words, the points $\omega_i(X)$ form an (n^2+2n+1)-dimensional simplex $Sp_1^{n^2+2n+1}$ with unit edge. Then we apply ω_i to all the other vertices of Sp_1^n. As a result, we obtain $n+1$ copies of the simplex $Sp_1^{n^2+2n+1}$. Among the n^2+2n+2 vertices $\omega_i(X)$, there are at least n^2+n+1 vertices whose colours are different from the colours of the vertices of Sp^n. The same is true for all the other n vertices of Sp_1^n. Hence, we have $n+1$ subsets in an (n^2+2n+2)-element set, each of cardinality n^2+n+1. By the Dirichlet principle, their intersection is nonempty.

We therefore have at least one rotation of the space, say ω_1, such that the colours of all points $\omega_1(X_1),\dots,\omega_1(X_{n+1})\in Sp_1^n$ are different from the colours of the points $Y_1,\dots,Y_{n+1}\in Sp^n$ and are different from each other. Once again we obtain two n-dimensional simplices on an n-dimensional sphere S_r^n whose colouring uses $2n+2$ colours. Hence, the bound $\pi^{n^2+3n+1}(S_r^n)\geqslant 2n+2$ holds. It remains to bound the radius from below, using the inequality $a>\frac{n}{2n+2}$ arising from the application of Lemma 3 (see Fig. 8):

$$\begin{aligned} &x^2+\frac{n}{2n+2}=(l-x)^2+a^2\\ &\quad\Longrightarrow\quad 2lx=l^2+a^2-\frac{n}{2n+2}=\frac{(n+1)^3-n((n+1)^2+1)}{(n+1)(2(n+1)^2+2)}+a^2\\ &\quad\Longrightarrow\quad 2lx>\frac{(n^2+n+1)}{(n+1)(2(n+1)^2+2)}+\frac{n^2}{(2n+2)^2}\\ &\qquad=\frac{(n^2+n+1)(2n+2)+n^2((n+1)^2+1)}{4(n+1)^2((n+1)^2+1)}=\frac{n^4+4n^3+6n^2+4n+2}{4(n+1)^2((n+1)^2+1)}. \end{aligned}\tag{5.7}$$

Substituting (5.7) into the formula for the radius, we obtain

$$r^2=\frac{(2lx)^2}{4l^2}+\frac{n}{2n+2}>\frac{(n^4+4n^3+6n^2+4n+2)^2}{32(n+1)^6((n+1)^2+1)}+\frac{n}{2n+2}.\tag{5.8}$$

5.3.4. *Proof of Theorem* 6. As in the proofs of Theorems 4 and 5, we consider two simplices Sp^n and Sp_1^n. By an application of Lemma 3 we achieve that the number of colours in the colouring of Sp^n is $n+1$, and the radius a of the sphere circumscribed around Sp^n is bigger than $\frac{n}{2n+2}$. As in Theorem 4, we require each vertex of Sp_1^n to be at distance one from some n vertices of the simplex Sp^n. However, this time the distance l between the planes α and α' must be at least $\frac{1}{2}$.

Let ω be a rotation of the space around the plane $\alpha\supset Sp^n$ such that $|\omega(X)X|=1$, $X\in Sp_1^n$. In any pair of vertices $(\omega(X),X)$ there is at least one vertex whose colour is different from the colour of any vertex of Sp^n. This is because each of the vertices $\omega(X)$, X is at distance one from n vertices of Sp^n, and the colour of the remaining vertex of Sp^n (which we denoted A_X) is different from the colour of one of the vertices $\omega(X)$, X. Therefore, one of the sets

$$\{\omega(X_i),\,i=1,\dots,n+1\},\qquad \{X_i,\,i=1,\dots,n+1\}$$

has in its colouring at least $\lceil\frac{n+1}{2}\rceil$ colours which are different from the colours of Sp^n. This implies the required bound. It remains to determine the conditions on the radius of the sphere under the given restrictions on a and l. For statement 1) we use the construction of Case 1 in subsection 5.3.1, and for statement 2) we use the construction of Case 2 in the same subsection. The remaining calculations are very similar to those of Theorem 4.

We first prove statement 1). We need to verify the inequality $l^2 \geqslant \frac{1}{4}$. In our case it reduces to the solution of the following quadratic inequality:

$$\frac{1}{4} \leqslant 1 - a^2 - \frac{n}{2n+2} + a\sqrt{\frac{2}{n(n+1)}} \quad \Longrightarrow \quad a^2 - a\sqrt{\frac{2}{n(n+1)}} - \frac{n+3}{4(n+1)} \leqslant 0$$
$$\Longrightarrow \quad \sqrt{\frac{1}{2n(n+1)}} - \sqrt{\frac{n+2}{4n}} \leqslant a \leqslant \sqrt{\frac{1}{2n(n+1)}} + \sqrt{\frac{n+2}{4n}}.$$

The left inequality is evidently satisfied, since $a > \frac{n}{2n+2}$. This implies the following restrictions on a:

$$\frac{n}{2n+2} < a \leqslant \sqrt{\frac{1}{2n(n+1)}} + \sqrt{\frac{n+2}{4n}}.$$

The left-hand side of the inequality above is less than $\frac{1}{2}$, and the right-hand side is bigger than $\frac{1}{2}$. Repeating the argument from Theorem 4 we obtain that, under the given restrictions, the value of r^2 grows with the growth of a. Therefore, the value of r is maximized when a is minimized, and vice versa. We obtain the following restrictions on r:

$$r^2 - \frac{n}{2n+2} = \frac{\left(\frac{1}{n+1} + a\sqrt{\frac{2}{n(n+1)}}\right)^2}{4l^2} > \frac{\left(\frac{1}{n+1} + \frac{n}{2n+2}\sqrt{\frac{2}{n(n+1)}}\right)^2}{4\left(1 - \left(\frac{n}{2n+2}\right)^2 - \frac{n}{2n+2} + \frac{n}{2n+2}\sqrt{\frac{2}{n(n+1)}}\right)}$$
$$\Longrightarrow \quad r^2 > \frac{\left(1 + \sqrt{\frac{n}{2(n+1)}}\right)^2}{n^2 + 6n + 4 + \sqrt{8n(n+1)}} + \frac{n}{2n+2}.$$

We note that in the case of the upper bound we have $l^2 = \frac{1}{4}$. Carrying out a similar calculation,

$$r^2 - \frac{n}{2n+2} = \frac{\left(\frac{1}{n+1} + a\sqrt{\frac{2}{n(n+1)}}\right)^2}{4l^2}$$
$$\leqslant \left(\frac{1}{n+1} + \left(\sqrt{\frac{1}{2n(n+1)}} + \sqrt{\frac{n+2}{4n}}\right)\sqrt{\frac{2}{n(n+1)}}\right)^2$$
$$\Longrightarrow \quad r^2 \leqslant \left(\frac{1}{n} + \frac{1}{n}\sqrt{\frac{n+2}{2n+2}}\right)^2 + \frac{n}{2n+2} = \frac{(\sqrt{n+2} + \sqrt{2n+2})^2 + n^3}{(2n+2)n^2}.$$

and taking the square root of the inequalities above, we obtain the bounds of statement 1).

Now we prove statement 2). Once again we need to verify the inequality $l^2 \geqslant \frac{1}{4}$. It reduces to the solution of the following quadratic inequality:

$$\frac{1}{4} \leqslant 1 - a^2 - \frac{n}{2n+2} - a\sqrt{\frac{2}{n(n+1)}} \quad \Longrightarrow \quad a^2 + a\sqrt{\frac{2}{n(n+1)}} - \frac{n+3}{4(n+1)} \leqslant 0$$
$$\Longrightarrow \quad -\sqrt{\frac{1}{2n(n+1)}} - \sqrt{\frac{n+2}{4n}} \leqslant a \leqslant \sqrt{\frac{n+2}{4n}} - \sqrt{\frac{1}{2n(n+1)}}.$$

The left inequality is evidently satisfied, and using the inequality $a > \frac{n}{2n+2}$ we obtain the following restrictions on a:

$$\frac{n}{2n+2} < a \leqslant \sqrt{\frac{n+2}{4n}} - \sqrt{\frac{1}{2n(n+1)}}.$$

We need to determine whether this double inequality has a solution. In the case $n = 1$ we have $\frac{1}{3} < a < \sqrt{\frac{3}{4}} - \frac{1}{2}$, $\sqrt{\frac{3}{4}} > \frac{5}{6}$, so a solution exists. We shall prove that a solution exists also for $n \geqslant 2$. We have

$$\frac{n}{2n+2} < \sqrt{\frac{n+2}{4n}} - \sqrt{\frac{1}{2n(n+1)}} \quad \Longleftrightarrow \quad \frac{n}{n+1} + \sqrt{\frac{2}{n(n+1)}} < \sqrt{\frac{n+2}{n}}. \tag{5.9}$$

Furthermore,

$$\frac{n}{n+1} + \sqrt{\frac{2}{n(n+1)}} < \sqrt{\frac{n}{n+2}} + \frac{\sqrt{2}}{n},$$

hence,

$$\left(\sqrt{\frac{n}{n+2}} + \frac{\sqrt{2}}{n} \leqslant \sqrt{\frac{n+2}{n}}\right) \quad \Longrightarrow \quad (5.9).$$

The inequality

$$\frac{\sqrt{2}}{n} \leqslant \sqrt{\frac{n+2}{n}}\left(1 - \frac{n}{n+2}\right) = \frac{2}{\sqrt{(n+2)n}}$$

is satisfied for any $n \geqslant 2$. Therefore, the double inequality on a has a solution for any n.

We need to study the behaviour of r^2 depending on a. To do this we differentiate the expression of r^2 with respect to a. We have

$$(r^2)' = \left(\frac{\left(\frac{1}{n+1} - a\sqrt{\frac{2}{n(n+1)}}\right)^2}{4\left(1 - a^2 - \frac{n}{2n+2} - a\sqrt{\frac{2}{n(n+1)}}\right)} + \frac{n}{2n+2}\right)' = \left(\frac{(c_1a - c_2)^2}{4(c_3 - a^2 - c_1a)}\right)',$$

where

$$c_1 = \sqrt{\frac{2}{n(n+1)}}, \qquad c_2 = \frac{1}{n+1}, \qquad c_3 = \frac{n+2}{2n+2}.$$

Continuing the formula for $(r^2)'$ above, we obtain

$$\begin{aligned}
(r^2)' &= \frac{2c_1(c_1a - c_2)(c_3 - a^2 - c_1a) + (c_1a - c_2)^2(2a + c_1)}{4(c_3 - a^2 - c_1a)^2} \\
&= \frac{(c_1a - c_2)(2c_1c_3 - 2c_1a^2 - 2c_1^2a + 2c_1a^2 - c_2c_1 - 2c_2a + c_1^2a)}{4(c_3 - a^2 - c_1a)^2} \\
&= \frac{(c_1a - c_2)(c_1(2c_3 - c_2) - c_1^2a - 2c_2a)}{4(c_3 - a^2 - c_1a)^2}.
\end{aligned}$$

We have

$$2c_3 - c_2 = \frac{n+2}{n+1} - \frac{1}{n+1} = 1, \qquad c_1^2 + 2c_2 = \frac{2}{n(n+1)} + \frac{2}{n+1} = \frac{2}{n},$$

hence,

$$\frac{c_1(2c_3 - c_2)}{c_1^2 + 2c_2} = \frac{n\sqrt{\frac{2}{n(n+1)}}}{2} = \sqrt{\frac{n}{2n+2}} = \frac{1}{\sqrt{\frac{2}{n(n+1)}}(n+1)} = \frac{c_2}{c_1}.$$

Therefore,

$$(r^2)' = -c_1(c_1^2 + 2c_2)\frac{(a - \frac{c_2}{c_1})^2}{4(c_3 - a^2 - c_1 a)^2} \leqslant 0.$$

In this case the maximum of r is achieved at $a = \frac{n}{2n+2}$, and the minimum is achieved at $a = \sqrt{\frac{n+2}{4n}} - \sqrt{\frac{1}{2n(n+1)}}$. For the latter value of a we have $l^2 = \frac{1}{4}$. We calculate

$$\begin{aligned} r^2 - \frac{n}{2n+2} &= \frac{\left(\frac{1}{n+1} - a\sqrt{\frac{2}{n(n+1)}}\right)^2}{4l^2} \\ &\geqslant \left(\frac{1}{n+1} + \left(\sqrt{\frac{1}{2n(n+1)}} - \sqrt{\frac{n+2}{4n}}\right)\sqrt{\frac{2}{n(n+1)}}\right)^2 \\ \implies \quad r^2 &\geqslant \left(\frac{1}{n} - \frac{1}{n}\sqrt{\frac{n+2}{2n+2}}\right)^2 + \frac{n}{2n+2} = \frac{(\sqrt{2n+2} - \sqrt{n+2})^2 + n^3}{(2n+2)n^2}, \\ r^2 - \frac{n}{2n+2} &= \frac{\left(\frac{1}{n+1} - a\sqrt{\frac{2}{n(n+1)}}\right)^2}{4l^2} < \frac{\left(\frac{1}{n+1} - \frac{n}{2n+2}\sqrt{\frac{2}{n(n+1)}}\right)^2}{4\left(1 - \left(\frac{n}{2n+2}\right)^2 - \frac{n}{2n+2} - \frac{n}{2n+2}\sqrt{\frac{2}{n(n+1)}}\right)} \\ \implies \quad r^2 &< \frac{\left(1 - \sqrt{\frac{n}{2(n+1)}}\right)^2}{n^2 + 6n + 4 - \sqrt{8n(n+1)}} + \frac{n}{2n+2}. \end{aligned}$$

Taking the square root, we obtain the required condition on r.

5.3.5. *Possible further results.* One might be able to formulate and prove Theorems 4 and 6 in a more general form. First, one could require that a k-dimensional simplex with unit edge is obtained under rotation of each vertex of Sp_1^n. In Theorem 6 we considered the case $k = 1$, and in Theorem 4 the case $k = n+1$. Furthermore, the assumption that the simplex Sp^n is coloured with $n+1$ colours could be replaced by the assumption that it is coloured with m colours. Then the minimal possible value of a clearly decreases. The corresponding bound for the speckledness would be as follows:

$$\pi^x(S^n) \geqslant n + m + 1 - \left\lfloor \frac{m}{k+1} \right\rfloor,$$

where $x = \max\{2m, n + k + 1\}$. This would be subject to some restrictions on the radius and the dimension n, which we do not specify here. The formula above arises as follows. First we apply Lemma 3 to construct an $(m-1)$-dimensional simplex coloured with m colours. Then we complete this simplex to a simplex of dimension n. After that we rotate the vertices of Sp_1^n along a k-dimensional sphere in such a way that each vertex together with its images form a k-dimensional simplex with unit edge. Fix a set of m vertices of different colours in Sp^n. Those vertices of Sp_1^n (and therefore, their images) which are at distance one from the m chosen vertices cannot be coloured with the colours of Sp^n, since all these colours are contained in the set of m vertices. Therefore, regardless of the rotation, at most m vertices of the simplex Sp_1^n can be coloured with one of the colours of Sp^n. We then proceed with an argument similar to that in the beginning of the proof of Theorem 6. Since each vertex and its k images under a rotation form a simplex with unit edge, we obtain that, under one of the rotations, at most $\lfloor \frac{m}{k+1} \rfloor$ vertices of this simplex will be coloured with the colours of Sp^n. This implies the bound above.

We shall not prove this theorem in its full generality, and restrict ourselves to a particular case. We shall prove Theorem 7.

Proof of Theorem 7. The construction is the same as that of Theorem 6. Naturally, the value l must be equal to $\frac{1}{2}$, as in our case each vertex of $Sp_1^n = Sp_1^2$ describes a 0-sphere S^0 under the rotation of the space around the plane α. The 0-sphere S^0 is simply a pair of points at distance $2l$ from each other,

$$|X\omega(X)| = 1, \qquad X \in Sp_1^2.$$

Furthermore, by Theorem 6, for $l = \frac{1}{2}$ the radius of the sphere under consideration is equal, depending on the case, to

$$r = \sqrt{\frac{(\sqrt{n+2} \pm \sqrt{2n+2})^2 + n^3}{(2n+2)n^2}}.$$

For $n = 2$ we obtain

$$r = \sqrt{\frac{3}{4} \pm \frac{1}{\sqrt{6}}}.$$

There is only one issue remaining. We cannot apply Lemma 3 in the case $n = 3$ to obtain a three-coloured triangle Sp^2. Instead, we proceed as follows. By application of Lemma 1 we choose a two-coloured segment AB of length $a\sqrt{3}$. Then we complete it to a regular triangle ABC. If the colour of the vertex C is different from the colours of A and B, then we proceed as in Theorem 6. Otherwise we may assume without loss of generality that the vertices A and C are of red colour, and B is of blue colour. Denote the vertices of Sp_1^2 by A_1, B_1, C_1; then $|C_1A| = |C_1B| = |A_1C| = |A_1B| = |B_1A| = |B_1C| = 1$. These conditions imply that none of the vertices A_1, $\omega(A_1)$, C_1, $\omega(C_1)$ is blue or red. On the other hand, one of the vertices B_1, $\omega(B_1)$ is not blue, and neither of them is red. Therefore, one of the triples $\{A_1, B_1, C_1\}$, $\{\omega(A_1), \omega(B_1), \omega(C_1)\}$ has colours different from blue and red. Assume that this is $\{A_1, B_1, C_1\}$. Then we need five colours to colour the points $\{A, B, C, A_1, B_1, C_1\}$, and all these points lie on S_r^2.

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Received 29/DEC/09 and 16/SEP/10
Translated by T. PANOV