The chromatic numbers of the normed spaces

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13.06.2011 – 17.06.2011, Budapest

Nelson–Hadwiger problem

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the chromatic number

what is the minimum number of colors which are needed to paint all the points on the plane so that any two points at distance 1 apart receive different colors?

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- Formally,

$$\chi(\mathbb{R}^d) = \min\{m \in \mathbb{N} : \mathbb{R}^d = H_1 \cup \ldots \cup H_m : \\ \forall i, \forall x, y \in H_i \quad |x - y| \neq 1\}.$$

Distance graph

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the *distance graph* $G = (V, E)$ *in* $\mathbb{R}^d$ is a graph with $V \subset \mathbb{R}^d$ and $E = \{(x, y), x, y \in \mathbb{R}^d, |x - y| = 1\}$.

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1951, Erdős, de Bruijn: If we accept the axiom of choice, then the chromatic number of the space is equal to the chromatic number of some finite distance graph in that space.

Some known results in small dimensions

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dim	4	5	6	7	8	9	10	11	12
$\chi \geq$	7	9	11	15	16	21	23	25	27

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- 2000, Raigorodskii, $\chi(\mathbb{R}^d) \geq (1,239.. + o(1))^d$

Generalizations and related problems

In the definition of the chromatic number instead of $\mathbb{R}^n$ with Euclidean metric we can consider an arbitrary space with an arbitrary metric.

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We also can consider colorings of the space of certain type.

- *Measurable chromatic number* $\chi^m(\mathbb{R}^2)$ (i.e. each color is a measurable set) is well-studied. We have $5 \leq \chi^m(\mathbb{R}^2) \leq 7$.

Asymptotical bounds

We denote by $\chi(\mathbb{R}^d_K, A)$ the chromatic number of the space with the norm induced by a convex centrally symmetric bounded body K and with the set A of forbidden distances.
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1. For $p = 2$ (classical case) we have $(1,239.. + o(1))^d \leq \chi(\mathbb{R}^d_2) = \chi(\mathbb{R}^d) \leq (3 + o(1))^d$. The upper bound is due to Larman, Rogers, 1972.

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2. We have

$$\chi(\mathbb{R}^d_p) \geq (1,207... + o(1))^d,$$
$$\chi(\mathbb{R}^d_\infty) = 2^d \quad \text{and}$$
$$\chi(\mathbb{R}^d_1) \geq (1,365... + o(1))^d.$$

Last result is due to Raigorodskii.

New results

Füredi and Kang in several works established different upper estimates on the $\chi(\mathbb{R}_p^d)$. The best one is

$$\chi(\mathbb{R}_K^d) \leq cd(\ln d)5^d. \qquad (2008)$$

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Theorem 1

We have

$$\chi(\mathbb{R}^d_K) \le \frac{(\ln d + \ln\ln d + \ln 4 + 1 + o(1))}{\ln\sqrt{2}} \cdot 4^d.$$

New results. l_p case.

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We have

$$\chi(\mathbb{R}_p^d) \le 2^{(1+c_p+\delta_d)d},$$

where $\delta_d \to 0$ as $d \to \infty$, and $c_p < 1$ as $p > 2$ and $c_p \to 0$ as $p \to \infty$.

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In particular, for $p(d) > \omega(d) d \ln\ln d$, $\omega(d) \to \infty$, we can obtain

$$\chi(\mathbb{R}_{p(d)}^d) \leq (\ln d + \ln\ln d + \ln 2 + 1 + o(1)) d 2^d = (2 + o(1))^d.$$

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Remind, that $\chi(\mathbb{R}_\infty^d) = 2^d$.

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4. Then we construct a suitable lattice packing and cover the space by its translates using the covering technique from item 1.

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Let $\mathbb{R}^d_K$ be a normed space.

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2. Let $l \geq 2$. Then $\chi(\mathbb{R}^d_p, A) \geq (b \cdot l)^d$, where $b = \frac{\sqrt[p']{2}}{2}$ and $p' = \max\{p, \frac{p}{p-1}\}$.

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3. Let $l \ge 2$. Then $\chi(\mathbb{R}^d, A) \ge (b \cdot l)^d$ where $b \approx 0,755 \cdot \sqrt{2}$.

Comment on Theorem 3. The chromatic number with multiple forbidden distances

We will limit ourself to the Euclidean case.
Let B be an arbitrary k-element set.

In general, we have an upper bound $\chi(\mathbb{R}^d, B) \le (3 + o(1))^{dk}$.

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$B_0 \subset A = [1, l]$ if $l = \sqrt{k}$, so, by Theorem 3,

$\chi(\mathbb{R}^d, B_0) \leq (2(\sqrt{k} + 1) + o(1))^d = (c_1' k)^{c_2' d}$ with some c_1', c_2'.

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Unfortunately, Theorem 3 does not give an improvement of the estimate from item 1 for an arbitrary k-element set B.

Comment on theorem 4. The gap between upper and lower bounds

1. In case of an arbitrary norm the gap between upper and lower bound in theorems 3 and 4 is

$(4\frac{l+1}{l} + \bar{o}(1))^d = (4 + \bar{o}(1))^d$ as $l, d \to \infty$.

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2. In the Euclidean case the gap is equal to

$(\frac{2}{b}^{\frac{l+1}{l}} + \bar{o}(1))^d \approx (1,87^{\frac{l+1}{l}} + \bar{o}(1))^d = (1,87 + \bar{o}(1))^d$ as $l, d \to \infty$.

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3. This is even better than the gap between upper and lower bounds for classical chromatic number. It is equal to

$$(\tfrac{3}{1.239} + \bar{o}(1))^d \approx (2.421 + \bar{o}(1))^d. \text{ as } d \to \infty.$$

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2. The technique used to obtain lower bounds is also based on a construction of some packing.
3. Additional ingredients are famous Kabatyanskiy – Levenshtein bound and Pichugov's bound on the radius of Jung's ball in $\mathbb{R}_p^d$.

Appendix. Proof of the lower bound

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We choose a point x painted in red. Then we draw three homothetic copies of K with center in x. One (K_1^x) is of radius 1 (we mean that it is the body $\|x\|_k \leq 1$), the other (K_l^x) is of radius l, the third $(K_{l/2}^x)$ is of radius $l/2$.

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There are no points of red color in $K^x_l \backslash K^x_1$.Then we choose another point not in K^x_l, colored in red etc.
Bodies $K^x_{l/2}$ form a packing in $\mathbb{R}^d_K$, meanwhile, all points of red color are contained in the union of K^x_1.

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Bodies $K_{l/2}^x$ form a packing in $\mathbb{R}^d_K$, meanwhile, all points of red color are contained in the union of K_1^x. The density of such union is not bigger than $Vol(K_1^x)/Vol(K_{l/2}^x) = (2/l)^d$. So, the chromatic number is not less than $(l/2)^d$.

The end

Thank You