On densest sets avoiding unit distance in spaces of small dimension

Andrey Kupavskii, Andrey Raigorodskii, Maria Titova

Lomonosov Moscow State University
Moscow, Russia

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Plan of the talk

1. The problem of finding densest sets avoiding unit distance.
   - History of the problem.
   - Brief exhibition of the method used to obtain the results.
2. Application of the results to the Ramsey-type problem concerning finding distance subgraphs of graphs in small-dimensional spaces.

Main definitions

Definition

The *upper density* of a Lebesgue measurable set $A \subseteq \mathbb{R}^n$ is

$$\overline{\delta}(A) = \overline{\lim_{r \to \infty}} \frac{V\left(A \cap B_n^{\mathbf{0}}(r)\right)}{V(B_n^{\mathbf{0}}(r))},$$

where $B_n^{\mathbf{0}}(r)$ is the ball of radius r centered at the origin, and $V(X)$ denotes volume of the set X.

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A subset S of the n-dimensional Euclidean space $\mathbb{R}^n$ *avoids unit distance*, if the distance between any two points in S never equals 1.

Definition

The *extreme density* of such set is

$$m_1(\mathbb{R}^n) = \sup\left\{\overline{\delta}(A) : A \subseteq \mathbb{R}^n \text{ is measurable and avoids unit distance}\right\}.$$

The extreme density $m_1(\mathbb{R}^n)$: motivation

Relation between $m_1(\mathbb{R}^n)$ and the *measurable chromatic number* $\chi^m(\mathbb{R}^n)$ of the Euclidean space:

Definition

The *chromatic number* $\chi(\mathbb{R}^n)$ is the minimum number of colors needed to paint all the points in $\mathbb{R}^n$ in such a way that any two points at unit distance apart receive different colors.

For $\chi^m(\mathbb{R}^n)$ it is additionally required that points receiving the same color form Lebesgue measurable sets.

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$$\chi^m(\mathbb{R}^n) \geq 1/m_1(\mathbb{R}^n) \Longrightarrow$$

upper bounds on $m_1(\mathbb{R}^n)$ are lower bounds on the measurable chromatic number $\chi^m(\mathbb{R}^n)$

Bounds on $\chi^m(\mathbb{R}^n)$ and $m_1(\mathbb{R}^n)$

Upper bounds on $m_1(\mathbb{R}^n), n \geq 2$, are due to F. M. de Oliveira Filho, F. Vallentin (2008). Bound on $\chi^m(\mathbb{R}^2)$ is due to K.J. Falconer (1981). The only case where lower bound on $\chi^m(\mathbb{R}^n)$ is better than $1/m_1(\mathbb{R}^n)$ is the case of the plane.

n	$\chi^m(\mathbb{R}^n) \geq$	$m_1(\mathbb{R}^n) \leq$
2	5	0.26841
3	7	0.16560
4	9	0.11293
5	14	0.07528
6	20	0.05157
7	28	0.03612
8	39	0.02579
9	54	0.01873
10	73	0.01380
11	97	0.01031
12	129	0.00780

Sets avoiding unit distance: motivation

- Study of $\chi(\mathbb{R}^n)$:

With some conditions on the structure of the set X the following theorem holds:

Theorem (Erdős, Rogers)

There exists a covering of the space by $\delta^{-1}(X) \cdot n \ln n(1 + o(1))$ copies of the set X.

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Corollary

If X is a set avoiding unit distance, then holds

$$\chi(\mathbb{R}^n) \leq \delta^{-1}(X) \cdot n \ln n(1 + o(1)).$$

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$$\chi(\mathbb{R}^n) \leq \delta^{-1}(X) \cdot n \ln n(1 + o(1)).$$

The best known asymptotic upper bound on the chromatic number of $\mathbb{R}^n$ is obtained using this theorem: $\chi(\mathbb{R}^n) \leq (3 + o(1))^n$.

Lower bounds on $m_1(\mathbb{R}^n)$: main result

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Theorem (K.,R.,T.)

The following inequalities hold:

$$m_1(\mathbb{R}^3) \geq 0.09877, \qquad m_1(\mathbb{R}^6) \geq 0.00806,$$
$$m_1(\mathbb{R}^4) \geq 0.04413, \qquad m_1(\mathbb{R}^7) \geq 0.00352,$$
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Note: sets constructed to obtain these bounds satisfy the conditions of Erdős – Rogers theorem.

Some definitions and notations

- A *lattice* L_n in $\mathbb{R}^n$ is the set

$$L_n = \left\{ \mathbf{a} \in \mathbb{R}^n \mid \mathbf{a} = \sum_{i=1}^{n} k_i \cdot \mathbf{v}_i,\ k_i \in \mathbb{Z} \right\}.$$

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- *The fundamental domain* Λ of L_n is the following set of points:

$$\Lambda = \left\{ \sum_{i=1}^{n} t_i \cdot \mathbf{v}_i,\, 0 \leq t_i < 1 \right\}.$$

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- *The determinant* $\det L_n$ *of the lattice* L_n is the volume of its fundamental domain.
- A collection $\mathcal{C} = \{C_1, C_2, ...\}$ of compact sets with nonempty interiors is said to form *a packing in* $\mathbb{R}^n$ if

$$\Omega = \bigcup_i C_i \subseteq \mathbb{R}^n$$

and no two sets in $\mathcal{C}$ have an interior point in common.
We also say that the set Ω is a packing in $\mathbb{R}^n$.

Some definitions and notations

- If the packing $\mathcal{C}$ in $\mathbb{R}^n$ consists of all translates of a particular Lebesgue measurable set $C \subset \mathbb{R}^n$ by vectors belonging to a given lattice L_n, i.e.

$$\mathcal{C} = \{C + \mathbf{a} \,|\, \mathbf{a} \in L_n\},$$

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- Note: $\delta(\Omega) = \overline{\delta}(\Omega)$ (according to the definition of the upper density).

The simplest way to obtain lower bounds on $m_1(\mathbb{R}^n)$

Let L_n be the lattice on which the densest packing $\Omega(r)$ of balls of radius r in $\mathbb{R}^n$ is realized.

$\Omega(r) = \bigcup\limits_{\mathbf{a} \in L_n} B_n^{\mathbf{a}}(r)$, where $B_n^{\mathbf{a}}(r)$ is the (open) ball in $\mathbb{R}^n$ of radius r centered in the lattice point $\mathbf{a} \in L_n$.

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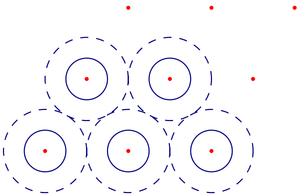
1. Decrease two times the radius of each ball.
2. New set avoids distance r.
3. Density of the new set is $\frac{\delta(\Omega(r))}{2^n}$.
   This value is the lower bound on $m_1(\mathbb{R}^n)$.



Croft's set

The density of the best packing on the plane: $\frac{\pi}{2\sqrt{3}} \approx 0.9069$.

Using the method described above we can obtain:

$$m_1(\mathbb{R}^2) \geq 0.2267.$$

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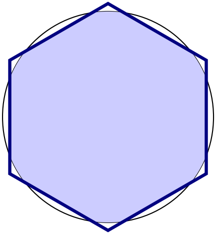
Croft made an example of the set, which density is equal to $0.2293...$, improving the previous bound.

A ,,tortoise":



Croft's set
is the union of ,,tortoises" centered
in points of a hexagonal lattice.

Our construction of densest sets in $\mathbb{R}^n$, $n = 3, \ldots, 8$

Definition

A *Voronoi polyhedron* $W_{L_n}^{\mathbf{a}}$ *of the lattice* L_n *in* $\mathbb{R}^n$ in the given lattice point $\mathbf{a}$ is a set of points of $\mathbb{R}^n$, which are at least as close to $\mathbf{a}$ as to any other point of the lattice:

$$W_{L_n}^{\mathbf{a}} = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{a}| \leq |\mathbf{x} - \mathbf{b}| \ \forall \ \mathbf{b} \in L_n\}.$$

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2. Put: $X^{\mathbf{a}}(r) := B^{\mathbf{a}}_n(r) \cap W^{\mathbf{a}}_{L_n}$ for every $\mathbf{a} \in L_n$.

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3. Do homothetic transformations of $X^{\mathbf{a}}(r)$ with homothety center in $\mathbf{a} \in L_n$ and such coefficient $k(r)$ that the union $\Omega(r)$ of obtained sets avoids unit distance.

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3. Do homothetic transformations of $X^{\mathbf{a}}(r)$ with homothety center in $\mathbf{a} \in L_n$ and such coefficient $k(r)$ that the union $\Omega(r)$ of obtained sets avoids unit distance.
4. Maximize the density of $\Omega(r)$ by choosing an appropriate radius r.

Comparison of the results

n	the biggest known density of the packing	known upper bound $m_1(\mathbb{R}^n)$ (Filho, Vallentin)	known lower bound $m_1(\mathbb{R}^n)$	new lower bound $m_1(\mathbb{R}^n)$
2	0.90689	0.26841	0.2293 (Croft)	—
3	0.74048	0.16560	0.09256	0.09877
4	0.61685	0.11293	0.03855	0.04413
5	0.46526	0.07528	0.01453	0.01833
6	0.37295	0.05157	0.00582	0.00806
7	0.29530	0.03612	0.00230	0.00352
8	0.25367	0.02579	0.00099	0.00165

Application to the problem on finding distance subgraphs of graphs in spaces of small dimension

Definition

A *unit distance graph in the n-dimensional Euclidean space* is an arbitrary graph $G = (V, E)$, whose set of vertices V is a subset of $\mathbb{R}^n$ and

$$E = \{(\mathbf{x}, \mathbf{y}) : \ \mathbf{x}, \mathbf{y} \in V, \ \ |\mathbf{x} - \mathbf{y}| = 1\}.$$

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Relation with the chromatic number $\chi(\mathbb{R}^n)$:

$$\chi(\mathbb{R}^n) = \chi(G), \text{ where } G = (\mathbb{R}^n, E^n), \ E^n = \{(\mathbf{x}, \mathbf{y}) : \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, |\mathbf{x} - \mathbf{y}| = 1\}.$$

Moreover, $\chi(\mathbb{R}^n) = \chi(H)$ for some finite distance graph H (according to Erdős – De Bruijn theorem).

Classical Ramsey number $R(s,t)$: bounds

Definition

For given $s, t \in \mathbb{N}$ the classical *Ramsey number* $R(s,t)$ is the minimum natural m such that for any graph $G = (V, E)$ on m vertices, either G contains an s-independent set or its complement $\bar{G}$ to the complete graph K_m contains a t-independent set.

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- $\left(\frac{1}{162} + o(1)\right) \frac{t^2}{\ln t} \le R(3,t) \le (1 + o(1)) \frac{t^2}{\ln t}$

  (Kim, 1995; Ajtai et al., 1980)

- $R(s,t) < \binom{s+t-2}{t-1}$

  (Erdős, Szekeres, 1935)

- $R(s,s) > \frac{1}{e\sqrt{2}} (1 + o(1))\, s\, 2^{\frac{s}{2}}$

  (Erdős, 1947, with random colorings)

- $R(s,s) > \frac{\sqrt{2}}{e} (1 + o(1))\, s\, 2^{\frac{s}{2}}$

  (application of Lovász Local Lemma)

Distance Ramsey number $R(s, s, n)$: bounds

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Distance Ramsey number $R_{\text{NEH}}(s, t, n)$ is the minimum natural m such that for any graph G on m vertices, either G contains an induced s-vertex subgraph isomorphic to a distance graph in $\mathbb{R}^n$ or its complement $\bar{G}$ contains an induced t-vertex subgraph isomorphic to a distance graph in $\mathbb{R}^n$.

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- $R_{\mathrm{NEH}}(s,s,n) \leq (n+1)\binom{2s-2(n+1)}{s-(n+1)}$ for $s \geq n+1$.

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- For $n = O(\ln s)$, there exists a constant $\gamma > 0$ such that the inequality applies

$$R_{\mathrm{NEH}}(s, s, n) \geq e^{\gamma \frac{s}{\ln^8 s}}.$$

- Given $s, n \in \mathbb{N}$: $R_{\mathrm{NEH}}(s, s, n) > \frac{\sqrt{2}}{4e}(1 + o(1))\, m\, 2^{\frac{m}{2}}$, where $m = \left[\frac{s}{\chi(\mathbb{R}^n)}\right]$

Best bounds when n is small

Theorem 1

There exists a positive constant c, such that

$$R_{\mathrm{NEH}}(s, s, 2) \geq 2^{\frac{s}{2} - c\, s^{\frac{1}{3}} \ln s}.$$

Theorem 2

There exists a positive constant c, such that

$$R_{\mathrm{NEH}}(s, s, 3) \geq 2^{\frac{s}{2} - c\, \beta(s) s^{\frac{1}{2}} \ln s},$$

where $\beta(s) = 2^{\alpha^2(s)}$, and $\alpha(s)$ is the Ackermann function.

Auxilliary claims

Claim 1

There exists such constant $c_2 > 0$ and such $m_2 \in \mathbb{N}$, that for all $m > m_2$ and for every distant graph $G = (V, E)$ in $\mathbb{R}^2$ with m vertices $|E| \leq c_2 m^{\frac{4}{3}}$.

Claim 2

There exists such constant $c_3 > 0$ and such $m_3 \in \mathbb{N}$, that for all $m > m_3$ and for every distant graph $G = (V, E)$ in $\mathbb{R}^3$ with m vertices $|E| \leq c_3 \beta(m) m^{\frac{3}{2}}$, where $\beta(m) = 2^{\alpha^2(m)}$, and $\alpha(m)$ is the Ackermann function.

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Claim 3

Every distance graph in $\mathbb{R}^n$ with m vertices has 2^n independent sets whose total cardinality is at least $[c_n m]$, where c_n is the corresponding constant from the main theorem.

New results

Theorem

The following inequalities hold:

$$R_{\text{NEH}}(s,s,n) \geq \frac{1}{e \cdot 2^{n+\frac{2^{n-1}-1}{2^n}}}(1+o(1))k2^{\frac{k}{n+1}}, \qquad \text{where} \quad k = 2^n\left[c_n s\right],$$

$$c_4 = 0.04413, \qquad c_7 = 0.00352,$$
$$c_5 = 0.01833, \qquad c_8 = 0.00165.$$
$$c_6 = 0.00806,$$

Thank you!