Counterexamples to Borsuk's conjecture on spheres of small radii

Kupavskiy A.B. Raigorodskiy A.M.

Department of Mechanics and Mathematics
Moscow State University
Moscow, Russia

28.06.2010 – 2.07.2010, Paris

Borsuk partition problem

- The following problem was posed by K. Borsuk in 1933: *is it true that any set $\Omega \subset \mathbb{R}^d$ having diameter 1 can be divided into some parts $\Omega_1, \ldots, \Omega_{d+1}$ whose diameters are strictly smaller than 1?*
- $$\operatorname{diam}\Omega = \sup_{\mathbf{x},\mathbf{y}\in\Omega} |\mathbf{x}-\mathbf{y}|$$
- By $f(\Omega)$ we denote the value
$$f(\Omega) = \min\{f : \ \Omega = \Omega_1 \cup \ldots \cup \Omega_f, \ \ \forall\, i \ \ \operatorname{diam}\Omega_i < \operatorname{diam}\Omega\}$$
and $f(d) = \max_{\Omega\subset\mathbb{R}^d,\ \operatorname{diam}\Omega=1} f(\Omega)$.
- Borsuk's problem: *is it true that always $f(d) = d+1$?*

Borsuk partition problem

- The following problem was posed by K. Borsuk in 1933: *is it true that any set $\Omega \subset \mathbb{R}^d$ having diameter 1 can be divided into some parts $\Omega_1, \ldots, \Omega_{d+1}$ whose diameters are strictly smaller than 1?*
-
$$\operatorname{diam} \Omega = \sup_{\mathbf{x},\mathbf{y} \in \Omega} |\mathbf{x} - \mathbf{y}|$$
- By $f(\Omega)$ we denote the value
$$f(\Omega) = \min\{f : \ \Omega = \Omega_1 \cup \ldots \cup \Omega_f, \quad \forall\, i \quad \operatorname{diam} \Omega_i < \operatorname{diam} \Omega\}$$
and $f(d) = \max_{\Omega \subset \mathbb{R}^d, \ \operatorname{diam} \Omega = 1} f(\Omega)$.
- Borsuk's problem: *is it true that always $f(d) = d + 1$?*

Borsuk partition problem

- The following problem was posed by K. Borsuk in 1933: *is it true that any set $\Omega \subset \mathbb{R}^d$ having diameter 1 can be divided into some parts $\Omega_1, \ldots, \Omega_{d+1}$ whose diameters are strictly smaller than 1?*
- $$\operatorname{diam} \Omega = \sup_{\mathbf{x},\mathbf{y} \in \Omega} |\mathbf{x} - \mathbf{y}|$$
- By $f(\Omega)$ we denote the value

  $$f(\Omega) = \min\{f : \ \Omega = \Omega_1 \cup \ldots \cup \Omega_f, \quad \forall\, i \quad \operatorname{diam} \Omega_i < \operatorname{diam} \Omega\}$$

  and $f(d) = \max_{\Omega \subset \mathbb{R}^d,\ \operatorname{diam} \Omega = 1} f(\Omega)$.
- Borsuk's problem: *is it true that always $f(d) = d + 1$?*

Borsuk partition problem

- The following problem was posed by K. Borsuk in 1933: *is it true that any set $\Omega \subset \mathbb{R}^d$ having diameter 1 can be divided into some parts $\Omega_1, \dots, \Omega_{d+1}$ whose diameters are strictly smaller than 1?*
- $$\operatorname{diam}\Omega = \sup_{\mathbf{x},\mathbf{y}\in\Omega} |\mathbf{x} - \mathbf{y}|$$
- By $f(\Omega)$ we denote the value

  $$f(\Omega) = \min\{f : \ \Omega = \Omega_1 \cup \dots \cup \Omega_f, \quad \forall\, i \quad \operatorname{diam}\Omega_i < \operatorname{diam}\Omega\}$$

  and $f(d) = \max_{\Omega\subset\mathbb{R}^d,\ \operatorname{diam}\Omega=1} f(\Omega)$.
- Borsuk's problem: *is it true that always $f(d) = d + 1$?*

History and some known results

1. 1946, H. Hadwiger, if Ω has smooth boundary, then $f(\Omega) \leq d+1$
2. 1993, J. Kahn and G. Kalai disproved the conjecture. They constructed a *finite* set of points in a very high dimension d that could not be decomposed into $d+1$ subsets of smaller diameter
3. Borsuk's conjecture is shown to be true for $d \leq 3$ and false for $d \geq 298$
4. $(1.2255... + o(1))^{\sqrt{d}} \leq f(d) \leq (1.224... + o(1))^d$.

History and some known results

1. 1946, H. Hadwiger, if Ω has smooth boundary, then $f(\Omega) \leq d + 1$
2. 1993, J. Kahn and G. Kalai disproved the conjecture. They constructed a *finite* set of points in a very high dimension d that could not be decomposed into $d + 1$ subsets of smaller diameter
3. Borsuk's conjecture is shown to be true for $d \leq 3$ and false for $d \geq 298$
4. $(1.2255... + o(1))^{\sqrt{d}} \leq f(d) \leq (1.224... + o(1))^d$.

History and some known results

1. 1946, H. Hadwiger, if Ω has smooth boundary, then $f(\Omega) \leq d+1$
2. 1993, J. Kahn and G. Kalai disproved the conjecture. They constructed a *finite* set of points in a very high dimension d that could not be decomposed into $d+1$ subsets of smaller diameter
3. Borsuk's conjecture is shown to be true for $d \leq 3$ and false for $d \geq 298$
4. $(1.2255... + o(1))^{\sqrt{d}} \leq f(d) \leq (1.224... + o(1))^d$.

History and some known results

1. 1946, H. Hadwiger, if Ω has smooth boundary, then $f(\Omega) \leq d + 1$
2. 1993, J. Kahn and G. Kalai disproved the conjecture. They constructed a *finite* set of points in a very high dimension d that could not be decomposed into $d + 1$ subsets of smaller diameter
3. Borsuk's conjecture is shown to be true for $d \leq 3$ and false for $d \geq 298$
4. $(1.2255... + o(1))^{\sqrt{d}} \leq f(d) \leq (1.224... + o(1))^d$.

Related problem and new theorem

- All known counterexamples to Borsuk's conjecture are always finite sets of points in $\mathbb{R}^d$ lying on spheres whose radii are close to $\frac{1}{\sqrt{2}}$
- It is quite natural, since, By Jung's theorem, any set in $\mathbb{R}^d$ having diameter 1 can be covered by a ball of radius $\sqrt{\frac{d}{2d+2}} \sim \frac{1}{\sqrt{2}}$
- **Theorem 1.** *For any $r > \frac{1}{2}$, there exists a $d_0 = d_0(r)$ such that for every $d \geq d_0$, one can find a set $\Omega \subset S_r^{d-1}$ which has diameter 1 and does not admit a partition into $d+1$ parts of smaller diameter.*

Related problem and new theorem

- All known counterexamples to Borsuk's conjecture are always finite sets of points in $\mathbb{R}^d$ lying on spheres whose radii are close to $\frac{1}{\sqrt{2}}$
- It is quite natural, since, By Jung's theorem, any set in $\mathbb{R}^d$ having diameter 1 can be covered by a ball of radius $\sqrt{\frac{d}{2d+2}} \sim \frac{1}{\sqrt{2}}$
- **Theorem 1.** *For any $r > \frac{1}{2}$, there exists a $d_0 = d_0(r)$ such that for every $d \geq d_0$, one can find a set $\Omega \subset S_r^{d-1}$ which has diameter 1 and does not admit a partition into $d+1$ parts of smaller diameter.*

Related problem and new theorem

- All known counterexamples to Borsuk's conjecture are always finite sets of points in $\mathbb{R}^d$ lying on spheres whose radii are close to $\frac{1}{\sqrt{2}}$
- It is quite natural, since, By Jung's theorem, any set in $\mathbb{R}^d$ having diameter 1 can be covered by a ball of radius $\sqrt{\frac{d}{2d+2}} \sim \frac{1}{\sqrt{2}}$
- **Theorem 1.** *For any $r > \frac{1}{2}$, there exists a $d_0 = d_0(r)$ such that for every $d \geq d_0$, one can find a set $\Omega \subset S_r^{d-1}$ which has diameter 1 and does not admit a partition into $d+1$ parts of smaller diameter.*

New results

- $f_r(d) = \max_{\Omega \subset S_r^{d-1},\ \mathrm{diam}\,\Omega = 1} f(\Omega)$.
- In these terms, Theorem 1 says that for any $r > \frac{1}{2}$, there exists a $d_0 = d_0(r)$ such that for every $d \geq d_0$, $f_r(d) > d + 1$

Theorem 2.

For any $r > \frac{1}{2}$, there exist numbers $k = k(r) \in \mathbb{N}$, $c = c(r) > 1$ and a function $\delta = \delta(d) = o(1)$ such that

$$f_r(d) \geq (c + \delta)^{\sqrt[2k]{d}}.$$

-

New results

- $f_r(d) = \max_{\Omega \subset S_r^{d-1},\ \mathrm{diam}\,\Omega = 1} f(\Omega)$.
- In these terms, Theorem 1 says that for any $r > \frac{1}{2}$, there exists a $d_0 = d_0(r)$ such that for every $d \geq d_0$, $f_r(d) > d + 1$

Theorem 2.

For any $r > \frac{1}{2}$, there exist numbers $k = k(r) \in \mathbb{N}$, $c = c(r) > 1$ and a function $\delta = \delta(d) = o(1)$ such that

$$f_r(d) \geq (c + \delta)^{\sqrt[2k]{d}}.$$

New results

- $f_r(d) = \max_{\Omega \subset S_r^{d-1},\ \mathrm{diam}\,\Omega = 1} f(\Omega)$.
- In these terms, Theorem 1 says that for any $r > \frac{1}{2}$, there exists a $d_0 = d_0(r)$ such that for every $d \geq d_0$, $f_r(d) > d + 1$

Theorem 2.

For any $r > \frac{1}{2}$, there exist numbers $k = k(r) \in \mathbb{N}$, $c = c(r) > 1$ and a function $\delta = \delta(d) = o(1)$ such that

$$f_r(d) \geq (c + \delta)^{\sqrt[2k]{d}}.$$

New results

Theorem 3.

Let $r = r(d) = \frac{1}{2} + \varphi(d)$, where $\varphi = o(1)$ and $\varphi(d) \geq c\frac{\ln\ln d}{\ln d}$ for all d and a large enough $c > 0$. Then, there exists a d_0 such that for $d \geq d_0$, $f_{r(d)}(d) > d + 1$.

Theorem 4.

Let $r = r(d) = \frac{1}{2} + \varphi(d)$, where $\varphi = O(1/d)$. Then, $f_r(d) \leq d + 1$.

New results

Theorem 3.

Let $r = r(d) = \frac{1}{2} + \varphi(d)$, where $\varphi = o(1)$ and $\varphi(d) \geq c\frac{\ln\ln d}{\ln d}$ for all d and a large enough $c > 0$. Then, there exists a d_0 such that for $d \geq d_0$, $f_{r(d)}(d) > d + 1$.

Theorem 4.

Let $r = r(d) = \frac{1}{2} + \varphi(d)$, where $\varphi = O(1/d)$. Then, $f_r(d) \leq d + 1$.

Linear-algebraic method

1. Consider the set

$$\Sigma = \{\mathbf{x} = (x_1, \ldots, x_n) : \forall\, i \;\; x_i \in \{-1, 1\},\; x_1 = 1,\; x_1 + \ldots + x_n = 0\}.$$

2. Let a be chosen in such a way that $p = \frac{a}{4} + \frac{n}{4}$ is a prime number and that $a \sim a_0 n$, $a_0 \in (0, 1)$
3. **Lemma.** *If $Q \subset \Sigma$ is such that $|Q| > \sum_{i=0}^{p-1} C_n^i$, then there exist $\mathbf{x}, \mathbf{y} \in Q$ with $(\mathbf{x}, \mathbf{y}) = -a$.*

Linear-algebraic method

1. Consider the set

$$\Sigma = \{\mathbf{x} = (x_1, \ldots, x_n) : \forall\, i \;\; x_i \in \{-1, 1\},\; x_1 = 1,\; x_1 + \ldots + x_n = 0\}.$$

2. Let a be chosen in such a way that $p = \frac{a}{4} + \frac{n}{4}$ is a prime number and that $a \sim a_0 n$, $a_0 \in (0, 1)$

3. **Lemma.** *If $Q \subset \Sigma$ is such that $|Q| > \sum_{i=0}^{p-1} C_n^i$, then there exist $\mathbf{x}, \mathbf{y} \in Q$ with $(\mathbf{x}, \mathbf{y}) = -a$.*

Linear-algebraic method

1. Consider the set

$$\Sigma = \{\mathbf{x} = (x_1, \ldots, x_n) : \forall\, i \;\; x_i \in \{-1, 1\},\; x_1 = 1,\; x_1 + \ldots + x_n = 0\}.$$

2. Let a be chosen in such a way that $p = \frac{a}{4} + \frac{n}{4}$ is a prime number and that $a \sim a_0 n$, $a_0 \in (0, 1)$

3. **Lemma.** *If $Q \subset \Sigma$ is such that $|Q| > \sum_{i=0}^{p-1} C_n^i$, then there exist $\mathbf{x}, \mathbf{y} \in Q$ with $(\mathbf{x}, \mathbf{y}) = -a$.*

Dual mapping. Definition

1. Let $\mathcal{A} = \{a_1, \ldots, a_w\}$ be the set of all possible $2k$-character words over the alphabet $X = \{1, \ldots, n\}$. Fix an $\mathbf{x} = (x_1, \ldots, x_n) \in \Sigma$. Consider

$$\mathbf{x}^{*2k} = \left(\mathbf{x}_{a_1}, \ldots, \mathbf{x}_{a_w}, \sqrt{2ka^{2k-1}}x_1, \ldots, \sqrt{2ka^{2k-1}}x_n\right),$$

where $\mathbf{x}_{a_j} = x_{i_1} \cdot \ldots \cdot x_{i_{2k}}$ if $a_j = i_1 \ldots i_{2k}$

2. The number of coordinates in any vector $\mathbf{x}^{*2k}$ equals $d = n^{2k} + n$. Put

$$\Omega' = \left\{\mathbf{x}^{*2k} : \mathbf{x} \in \Sigma\right\}.$$

Dual mapping. Definition

1. Let $\mathcal{A} = \{a_1, \ldots, a_w\}$ be the set of all possible $2k$-character words over the alphabet $X = \{1, \ldots, n\}$. Fix an $\mathbf{x} = (x_1, \ldots, x_n) \in \Sigma$. Consider

$$\mathbf{x}^{*2k} = \left(\mathbf{x}_{a_1}, \ldots, \mathbf{x}_{a_w}, \sqrt{2ka^{2k-1}}x_1, \ldots, \sqrt{2ka^{2k-1}}x_n\right),$$

   where $\mathbf{x}_{a_j} = x_{i_1} \cdot \ldots \cdot x_{i_{2k}}$ if $a_j = i_1 \ldots i_{2k}$

2. The number of coordinates in any vector $\mathbf{x}^{*2k}$ equals $d = n^{2k} + n$. Put

$$\Omega' = \left\{\mathbf{x}^{*2k} : \ \mathbf{x} \in \Sigma\right\}.$$

Dual mapping. Properties of Ω'

1. It is obvious that there is a bijection between Σ and Ω'
2. $$\left(\mathbf{x}^{*2k}, \mathbf{y}^{*2k}\right) = (\mathbf{x}, \mathbf{y})^{2k} + 2ka^{2k-1}(\mathbf{x}, \mathbf{y}).$$
3. The minimum of the form $\left(\mathbf{x}^{*2k}, \mathbf{y}^{*2k}\right)$, which refers to the diameter of Ω', is attained on those and only those pairs of vectors $\mathbf{x}, \mathbf{y} \in \Sigma$ whose scalar product equals $-a$
4. Using the lemma, we obtain that

$$f(\Omega') \geq \frac{|\Omega'|}{\sum_{i=0}^{p-1} C_n^i} = \frac{C_{n-1}^{\frac{n}{2}-1}}{\sum_{i=0}^{p-1} C_n^i} = (c+\delta)^n = (c+\delta)^{\sqrt[2k]{d}},$$

where $c = const > 1$, $\delta \to 0$

Dual mapping. Properties of Ω'

1. It is obvious that there is a bijection between Σ and Ω'
2. $$\left(\mathbf{x}^{*2k}, \mathbf{y}^{*2k}\right) = (\mathbf{x}, \mathbf{y})^{2k} + 2ka^{2k-1}(\mathbf{x}, \mathbf{y}).$$
3. The minimum of the form $\left(\mathbf{x}^{*2k}, \mathbf{y}^{*2k}\right)$, which refers to the diameter of Ω', is attained on those and only those pairs of vectors $\mathbf{x}, \mathbf{y} \in \Sigma$ whose scalar product equals $-a$
4. Using the lemma, we obtain that

$$f(\Omega') \geq \frac{|\Omega'|}{\sum\limits_{i=0}^{p-1} C_n^i} = \frac{C_{n-1}^{\frac{n}{2}-1}}{\sum\limits_{i=0}^{p-1} C_n^i} = (c+\delta)^n = (c+\delta)^{\sqrt[2k]{d}},$$

where $c = const > 1$, $\delta \to 0$

Dual mapping. Properties of Ω'

1. It is obvious that there is a bijection between Σ and Ω'
2. $$\left(\mathbf{x}^{*2k}, \mathbf{y}^{*2k}\right) = (\mathbf{x}, \mathbf{y})^{2k} + 2ka^{2k-1}(\mathbf{x}, \mathbf{y}).$$
3. The minimum of the form $\left(\mathbf{x}^{*2k}, \mathbf{y}^{*2k}\right)$, which refers to the diameter of Ω', is attained on those and only those pairs of vectors $\mathbf{x}, \mathbf{y} \in \Sigma$ whose scalar product equals $-a$
4. Using the lemma, we obtain that

$$f(\Omega') \geq \frac{|\Omega'|}{\sum_{i=0}^{p-1} C_n^i} = \frac{C_{n-1}^{\frac{n}{2}-1}}{\sum_{i=0}^{p-1} C_n^i} = (c+\delta)^n = (c+\delta)^{\sqrt[2k]{d}},$$

where $c = const > 1$, $\delta \to 0$

Dual mapping. Properties of Ω'

1. It is obvious that there is a bijection between Σ and Ω'
2. $$\left(\mathbf{x}^{*2k}, \mathbf{y}^{*2k}\right) = (\mathbf{x}, \mathbf{y})^{2k} + 2ka^{2k-1}(\mathbf{x}, \mathbf{y}).$$
3. The minimum of the form $\left(\mathbf{x}^{*2k}, \mathbf{y}^{*2k}\right)$, which refers to the diameter of Ω', is attained on those and only those pairs of vectors $\mathbf{x}, \mathbf{y} \in \Sigma$ whose scalar product equals $-a$
4. Using the lemma, we obtain that

$$f(\Omega') \geq \frac{|\Omega'|}{\sum\limits_{i=0}^{p-1} C_n^i} = \frac{C_{n-1}^{\frac{n}{2}-1}}{\sum\limits_{i=0}^{p-1} C_n^i} = (c+\delta)^n = (c+\delta)^{\sqrt[2k]{d}},$$

where $c = const > 1$, $\delta \to 0$

The diameter of Ω'

1. It is clear that Ω' lies on the sphere S_{ρ}^{d-1}, where
$$\rho^2 = \left(\mathbf{x}^{*2k}, \mathbf{x}^{*2k}\right) = n^{2k} + 2ka^{2k-1}n.$$
2. $\operatorname{diam}^2 \Omega' = 2n^{2k} + 4ka^{2k-1}n + (4k-2)a^{2k}$.
3. Compressing Ω' so that a new set Ω'' has diameter 1, we see that $\Omega'' \subset S_{r'}^{d-1}$ with
$$(r')^2 = \frac{n^{2k} + 2ka^{2k-1}n}{2n^{2k} + 4ka^{2k-1}n + (4k-2)a^{2k}}.$$
4. Remind, that $a = a_0 n$, and $a_0 \in (0,1)$ is arbitrary, so, when $a_0 \to 1$, $(r')^2 \to \frac{2k+1}{8k}$
5. for arbitrary $r > r' > \sqrt{\frac{2k+1}{8k}}$ we can embed $S_{r'}^{d'-1}$ into S_r^d and receive a counterexample on the sphere of desired radius

The diameter of Ω'

1. It is clear that Ω' lies on the sphere S_ρ^{d-1}, where
$$\rho^2 = \left(\mathbf{x}^{*2k}, \mathbf{x}^{*2k}\right) = n^{2k} + 2ka^{2k-1}n.$$
2. $\operatorname{diam}^2 \Omega' = 2n^{2k} + 4ka^{2k-1}n + (4k-2)a^{2k}$.
3. Compressing Ω' so that a new set Ω'' has diameter 1, we see that $\Omega'' \subset S_{r'}^{d-1}$ with
$$(r')^2 = \frac{n^{2k} + 2ka^{2k-1}n}{2n^{2k} + 4ka^{2k-1}n + (4k-2)a^{2k}}.$$
4. Remind, that $a = a_0 n$, and $a_0 \in (0,1)$ is arbitrary, so, when $a_0 \to 1$, $(r')^2 \to \frac{2k+1}{8k}$
5. for arbitrary $r > r' > \sqrt{\frac{2k+1}{8k}}$ we can embed $S_{r'}^{d'-1}$ into S_r^d and receive a counterexample on the sphere of desired radius

The diameter of Ω'

1. It is clear that Ω' lies on the sphere S_{ρ}^{d-1}, where

$$\rho^2 = \left(\mathbf{x}^{*2k}, \mathbf{x}^{*2k}\right) = n^{2k} + 2ka^{2k-1}n.$$

2. $\operatorname{diam}^2 \Omega' = 2n^{2k} + 4ka^{2k-1}n + (4k-2)a^{2k}$.
3. Compressing Ω' so that a new set Ω'' has diameter 1, we see that $\Omega'' \subset S_{r'}^{d-1}$ with

$$(r')^2 = \frac{n^{2k} + 2ka^{2k-1}n}{2n^{2k} + 4ka^{2k-1}n + (4k-2)a^{2k}}.$$

4. Remind, that $a = a_0 n$, and $a_0 \in (0,1)$ is arbitrary, so, when $a_0 \to 1$, $(r')^2 \to \frac{2k+1}{8k}$
5. for arbitrary $r > r' > \sqrt{\frac{2k+1}{8k}}$ we can embed $S_{r'}^{d'-1}$ into S_r^d and receive a counterexample on the sphere of desired radius

The diameter of Ω'

1. It is clear that Ω' lies on the sphere S_ρ^{d-1}, where
$$\rho^2 = \left(\mathbf{x}^{*2k}, \mathbf{x}^{*2k}\right) = n^{2k} + 2ka^{2k-1}n.$$
2. $\operatorname{diam}^2 \Omega' = 2n^{2k} + 4ka^{2k-1}n + (4k-2)a^{2k}$.
3. Compressing Ω' so that a new set Ω'' has diameter 1, we see that $\Omega'' \subset S_{r'}^{d-1}$ with
$$(r')^2 = \frac{n^{2k} + 2ka^{2k-1}n}{2n^{2k} + 4ka^{2k-1}n + (4k-2)a^{2k}}.$$
4. Remind, that $a = a_0 n$, and $a_0 \in (0,1)$ is arbitrary, so, when $a_0 \to 1$, $(r')^2 \to \frac{2k+1}{8k}$
5. for arbitrary $r > r' > \sqrt{\frac{2k+1}{8k}}$ we can embed $S_{r'}^{d'-1}$ into S_r^d and receive a counterexample on the sphere of desired radius

The diameter of Ω'

1. It is clear that Ω' lies on the sphere S_ρ^{d-1}, where

$$\rho^2 = \left(\mathbf{x}^{*2k}, \mathbf{x}^{*2k}\right) = n^{2k} + 2ka^{2k-1}n.$$

2. $\operatorname{diam}^2 \Omega' = 2n^{2k} + 4ka^{2k-1}n + (4k-2)a^{2k}$.
3. Compressing Ω' so that a new set Ω'' has diameter 1, we see that $\Omega'' \subset S_{r'}^{d-1}$ with

$$(r')^2 = \frac{n^{2k} + 2ka^{2k-1}n}{2n^{2k} + 4ka^{2k-1}n + (4k-2)a^{2k}}.$$

4. Remind, that $a = a_0 n$, and $a_0 \in (0,1)$ is arbitrary, so, when $a_0 \to 1$, $(r')^2 \to \frac{2k+1}{8k}$
5. for arbitrary $r > r' > \sqrt{\frac{2k+1}{8k}}$ we can embed $S_{r'}^{d'-1}$ into S_r^d and receive a counterexample on the sphere of desired radius

Remarks

- To obtain a counterexample on the sphere of the radius r we first choose k so, that $r > \sqrt{\frac{2k+1}{8k}}$, then we choose a_0 close enough to 1, then we choose a.
- Proof of the theorem 3 is based on the same ideas, but it is much more delicate because of the complicated optimization
- to prove theorem 4, we divide the sphere S_r^{d-1} into $d+1$ parts by inscribing into it a regular simplex

Remarks

- To obtain a counterexample on the sphere of the radius r we first choose k so, that $r > \sqrt{\frac{2k+1}{8k}}$, then we choose a_0 close enough to 1, then we choose a.
- Proof of the theorem 3 is based on the same ideas, but it is much more delicate because of the complicated optimization
- to prove theorem 4, we divide the sphere S_r^{d-1} into $d+1$ parts by inscribing into it a regular simplex

Remarks

- To obtain a counterexample on the sphere of the radius r we first choose k so, that $r > \sqrt{\frac{2k+1}{8k}}$, then we choose a_0 close enough to 1, then we choose a.
- Proof of the theorem 3 is based on the same ideas, but it is much more delicate because of the complicated optimization
- to prove theorem 4, we divide the sphere S_r^{d-1} into $d+1$ parts by inscribing into it a regular simplex

Remarks. Optimum

- If we fix $2k$ – an even natural number, we can prove that $f_r(d) \geq (c+\delta)^{\sqrt[2k]{d}}$ only for $r > \sqrt{\frac{2k+1}{8k}}$. In some sense it is the the best possible bound.
- Remind that after dual mapping the scalar product in Σ transformed into a polynomial from the scalar product in Ω':

$$(\mathbf{x},\mathbf{y}) \longrightarrow \left(\mathbf{x}^{*2k},\mathbf{y}^{*2k}\right) = (\mathbf{x},\mathbf{y})^{2k} + 2ka^{2k-1}(\mathbf{x},\mathbf{y}).$$

- We proved, that this mapping is optimal among all mappings f of a certain type:

$$(\mathbf{x},\mathbf{y}) \longrightarrow (f(\mathbf{x}),f(\mathbf{y})) = Pol((\mathbf{x},\mathbf{y})),$$

  where $Pol(t)$ is an arbitrary polynomial of degree $2k$
- We also proved, that it is nonoptimal to use polynomials of even degree

Remarks. Optimum

- If we fix $2k$ – an even natural number, we can prove that $f_r(d) \geq (c+\delta)^{\sqrt[2k]{d}}$ only for $r > \sqrt{\frac{2k+1}{8k}}$. In some sense it is the the best possible bound.
- Remind that after dual mapping the scalar product in Σ transformed into a polynomial from the scalar product in Ω':

$$(\mathbf{x}, \mathbf{y}) \longrightarrow \left(\mathbf{x}^{*2k}, \mathbf{y}^{*2k}\right) = (\mathbf{x}, \mathbf{y})^{2k} + 2ka^{2k-1}(\mathbf{x}, \mathbf{y}).$$

- We proved, that this mapping is optimal among all mappings f of a certain type:

$$(\mathbf{x}, \mathbf{y}) \longrightarrow (f(\mathbf{x}), f(\mathbf{y})) = Pol((\mathbf{x}, \mathbf{y})),$$

where $Pol(t)$ is an arbitrary polynomial of degree $2k$
- We also proved, that it is nonoptimal to use polynomials of even degree

Remarks. Optimum

- If we fix $2k$ – an even natural number, we can prove that $f_r(d) \geq (c+\delta)^{\sqrt[2k]{d}}$ only for $r > \sqrt{\frac{2k+1}{8k}}$. In some sense it is the the best possible bound.
- Remind that after dual mapping the scalar product in Σ transformed into a polynomial from the scalar product in Ω':

$$(\mathbf{x}, \mathbf{y}) \longrightarrow \left(\mathbf{x}^{*2k}, \mathbf{y}^{*2k}\right) = (\mathbf{x}, \mathbf{y})^{2k} + 2ka^{2k-1}(\mathbf{x}, \mathbf{y}).$$

- We proved, that this mapping is optimal among all mappings f of a certain type:

$$(\mathbf{x}, \mathbf{y}) \longrightarrow (f(\mathbf{x}), f(\mathbf{y})) = Pol((\mathbf{x}, \mathbf{y})),$$

where $Pol(t)$ is an arbitrary polynomial of degree $2k$
- We also proved, that it is nonoptimal to use polynomials of even degree

Remarks. Optimum

- If we fix $2k$ – an even natural number, we can prove that $f_r(d) \geq (c+\delta)^{\sqrt[2k]{d}}$ only for $r > \sqrt{\frac{2k+1}{8k}}$. In some sense it is the the best possible bound.
- Remind that after dual mapping the scalar product in Σ transformed into a polynomial from the scalar product in Ω':

$$(\mathbf{x}, \mathbf{y}) \longrightarrow \left(\mathbf{x}^{*2k}, \mathbf{y}^{*2k}\right) = (\mathbf{x}, \mathbf{y})^{2k} + 2ka^{2k-1}(\mathbf{x}, \mathbf{y}).$$

- We proved, that this mapping is optimal among all mappings f of a certain type:

$$(\mathbf{x}, \mathbf{y}) \longrightarrow (f(\mathbf{x}), f(\mathbf{y})) = Pol((\mathbf{x}, \mathbf{y})),$$

  where $Pol(t)$ is an arbitrary polynomial of degree $2k$
- We also proved, that it is nonoptimal to use polynomials of even degree

The end

Thank You