ISSN 0001-4346, Mathematical Notes, 2010, Vol. 87, No. 2, pp. 218–229. © Pleiades Publishing, Ltd., 2010.
Original Russian Text © A. B. Kupavskii, A. M. Raigorodskii, 2010, published in Matematicheskie Zametki, 2010, Vol. 87, No. 2, pp. 233–245.

Partition of Three-Dimensional Sets into Five Parts of Smaller Diameter

A. B. Kupavskii* **and A. M. Raigorodskii****

Moscow State University

Received June 4, 2008

Abstract—The classical Borsuk problem on partitioning sets into pieces of smaller diameter is considered. A new upper bound for

$$d_5^3 = \sup_{\Phi\subset\mathbb{R}^3,\,\mathrm{diam}\,\Phi=1} \inf\{x \ge 0 : \Phi = \Phi_1 \cup \Phi_2 \cup \dots \cup \Phi_5,\ \mathrm{diam}\,\Phi_i \le x\}$$

is given, which improves the previous bound obtained by Lassak in 1982.

DOI: 10.1134/S0001434610010281

Key words: *Borsuk's problem, partition of 3D sets, diameter of a set, Lassak's bound, Gale's conjecture, Jung's ball, Helly's theorem, isometry.*

1. STATEMENT OF THE PROBLEM AND RESULTS

This study is related to the classical Borsuk problem about partitioning sets into pieces of smaller diameter (see [1]–[8]). The setting which we consider in this paper was studied in, e.g., [9]–[11].

Let k and n be positive integers, and let Φ be any (bounded non-one-point) set in $\mathbb{R}^n$. We define the diameter of Φ as

$$\mathrm{diam}\,\Phi = \sup_{X,Y\in\Phi} \rho(X,Y),$$

where $\rho(X,Y)$ is the standard Euclidean metric, and consider the functions

$$d_k^n(\Phi) = \inf\{x \ge 0 : \Phi = \Phi_1 \cup \Phi_2 \cup \dots \cup \Phi_k,\ \mathrm{diam}\,\Phi_i \le x\}, \qquad d_k^n = \sup_{\Phi,\,\mathrm{diam}\,\Phi=1} d_k^n(\Phi).$$

The behavior of the functions $d_k^n(\Phi)$ and d_k^n has been studied fairly well for $n \le 2$ (see [9]–[11]), and we do not consider this case in this paper. At the same time, there are a number of deep results on the behavior of these functions for sufficiently large n and on their asymptotics as $n \to \infty$ (see [1]–[6]). However, virtually nothing is known for, say, $n = 3$.

It is fairly easy to understand that

$$d_1^3 = d_2^3 = d_3^3 = 1$$

(see, e.g., [8]). The proof of the inequality

$$d_4^3 \ge \sqrt{\frac{3+\sqrt{3}}{6}}$$

is somewhat more involved (see [8]). In 1953, Gale conjectured that this inequality cannot be improved (see [12]). This conjecture has not been proved or disproved so far. It is only known that $d_4^3 \le 0.98$ (see [8], [13]–[15]).

*E-mail: kupavskii@yandex.ru
**E-mail: mraigor@yandex.ru

The problem of evaluating d_5^3 is even farther from being solved.

In 1982, Lassak proved that

$$d^n_{2^{n-1}+1} \le \sqrt{\frac{4n^2+\sqrt{8n^2+1}-1}{4n^2+4n}}$$

(see [16]). As a corollary, for $n=3$, we obtain

$$d_5^3 \le \sqrt{\frac{35+\sqrt{73}}{48}} = 0.9524\dots .$$

The main result of this paper is an improvement of Lassak's bound.

Theorem 1. *The following inequality holds*:

$$d_5^3 \le 0.9425.$$

In the next section, we describe a general approach to obtaining bounds similar to Lassak's bound and to be found in Theorem 1.

2. GENERAL APPROACH

We say that a set U is a *universal cover* in $\mathbb{R}^n$ if, for any $\Phi \subset \mathbb{R}^n$ with $\operatorname{diam}\Phi = 1$, there exists an isometry O of space under which

$$O(U) \supseteq \Phi.$$

Thus, as early as in 1901, H. Jung proved that a ball of radius $r=\sqrt{n/(2n+2)}$ is a universal cover in $\mathbb{R}^n$. In what follows, we refer to such a ball as the *Jung ball* (see [17]).

To prove the bound given in Sec. 1, Lassak used the universal cover U in $\mathbb{R}^n$ with is the intersection of two balls, the Jung ball

$$B = \{X=(x_1,\dots,x_n) : x_1^2+\dots+x_n^2 \le r^2\}$$

and a ball of radius 1 centered at any point on the boundary of B:

$$U = B \cap \{X=(x_1,\dots,x_n) : (x_1-r)^2+x_2^2+\dots+x_n^2 \le 1\}.$$

We call a system of sets $\mathscr{U}$ a *universal cover system* (UCS) in $\mathbb{R}^n$ if, for any $\Phi \subset \mathbb{R}^n$ with $\operatorname{diam}\Phi = 1$, there exists a $U \in \mathscr{U}$ and an isometry O of space for which

$$O(U) \supseteq \Phi.$$

Clearly, a universal cover is a special case of a UCS. At the same time, a UPS may be infinite (see [18]). Anyway, it is easy to show that, whatever a UCS $\mathscr{U}$, we have (see [11])

$$d_k^n \le \sup_{U\in\mathscr{U}} d_k^n(U). \tag{1}$$

In the next section, we describe the UCSs which we use to prove Theorem 1. More details about UCS theory can be found in [19], [20].

3. CONSTRUCTION OF UNIVERSAL COVER SYSTEMS

Let

$$r = \sqrt{\frac{3}{2\cdot 3+2}} = \sqrt{\frac{3}{8}} = 0.612\dots$$

be the radius of the Jung ball in $\mathbb{R}^3$. Take any number d in the interval $[0.5, r]$. Consider the balls

$$D = D(d) = \{X = (x,y,z) : x^2+y^2+z^2 \le d^2\}$$

and

$$D_1 = \{X = (x,y,z) : (x-d)^2+y^2+z^2 \le 1\}.$$

We denote the corresponding spheres by S and S_1, respectively.

Take a point $\mathscr{O} = (a, 0, c)$ in the set

$$S \cap D_1 \cap \{X = (x,y,z) : x \le 0\} \cap \{X = (x,y,z) : y = 0\} \cap \{X = (x,y,z) : z \le 0\}.$$

The point $\mathscr{O}$ belongs to the closed arc $L = L(d)$ of the great circle

$$S \cap \{X = (x,y,z) : y = 0\}$$

whose "right" endpoint belongs to the negative part of the z axis and the "left" endpoint is at distance 1 from the center of the ball D_1 (that is, from the point $(d,0,0)$); see Fig. 1. We set

$$D_2 = D_2(\mathscr{O}) = \{X = (x,y,z) : (x-a)^2+y^2+(z-c)^2 \le 1\}.$$

This is the ball of radius 1 centered at $\mathscr{O}$. We also set $S_2 = \partial D_2$.

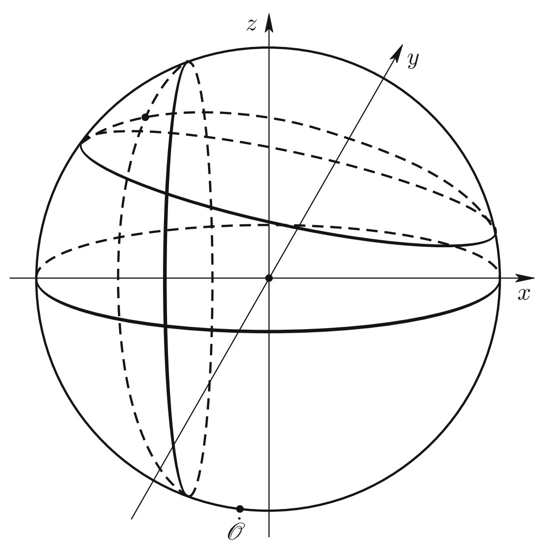

Fig. 1.

Finally, let

$$\mathscr{U} = \{U(d,\mathscr{O}) : d \in [0.5, r],\ \mathscr{O} \in L(d)\} = \bigcup_{d\in[0.5,r]} \bigcup_{\mathscr{O}\in L(d)} \{U(d,\mathscr{O})\}$$

be the family of sets

$$U(d,\mathscr{O}) = D \cap D_1 \cap D_2$$

having the cardinality of the continuum.

Clearly, the construction described above generalizes that of Jung and Lassak (see Sec. 2). However, what we consider is not a universal cover.

Lemma 1. *The family $\mathscr{U}$ forms a UCS in $\mathbb{R}^3$.*

A similar assertion was stated in the brief communication [18]. We prove the lemma below.

Proof of Lemma 1. Take a set Φ of diameter 1 in $\mathbb{R}^3$. Of course, we can assume without loss of generality that Φ is closed. Let d denote the minimal number for which Φ is covered by a copy of the ball $D = D(d)$ (that is, $\Phi \subseteq O(D)$ for suitable O). In what follows, we write simply "D covers Φ", meaning the presence of the corresponding isometry. Note that the ball D is well defined, because, surely, $d \in [0.5, r]$ for any Φ.

The well-known Helly theorem (see [17]) and the fact that Φ is closed imply the existence of a set $\{z_1, \dots, z_i\} \subseteq \Phi$ consisting of at most four points which cannot be covered by a ball of radius less than d but can be covered by D. In other words, D is a minimal ball containing $z_1, \dots, z_i$; clearly, this implies $|\{z_1, \dots, z_i\} \cap S| \geq 2$.

Without loss of generality (up to isometry), we can assume that $z_1 = (d, 0, 0) \in S$. Since the distance from any point of the set Φ to z_1 does not exceed 1, it follows that the ball D_1 covers Φ.

Clearly, at least one point from $\{z_1, \dots, z_i\} \cap S$ must belong to the half-space $\{X = (x, y, z) : x \leq 0\}$ (otherwise, the ball D is not minimal for $\{z_1, \dots, z_i\}$). Again, we can assume without loss of generality that this point coincides with some $\mathscr{O} \in L(d)$. Hence $D_2 = D_2(\mathscr{O})$ covers Φ.

We have found an isometry O, a number $d \in [0.5, r]$, and a point $\mathscr{O} \in L(d)$ for which

$$O(\Phi) \subseteq D(d) \cap D_1 \cap D_2(\mathscr{O}) = U(d, \mathscr{O}) \in \mathscr{U}.$$

This proves the lemma. □

4. CONSTRUCTION OF A PARTITION

We want to apply inequality (1) and Lemma 1. To this end, we partition each $U \in \mathscr{U}$ into five parts. Take U (and, therefore, d and $\mathscr{O}$) and choose any $\varepsilon, \delta \in (0, d)$. We denote the required parts by

$$P_1 = P_1(U, \varepsilon, \delta), \quad \dots, \quad P_5 = P_5(U, \varepsilon, \delta);$$

we set

$$P_1 = U \cap \{X = (x, y, z) : x \geq d - \varepsilon\},$$
$$P_2 = U \cap \{X = (x, y, z) : x \leq d - \varepsilon\} \cap \{X = (x, y, z) : z \geq -\delta\} \cap \{X = (x, y, z) : y \leq 0\},$$
$$P_3 = U \cap \{X = (x, y, z) : x \leq d - \varepsilon\} \cap \{X = (x, y, z) : z \leq -\delta\} \cap \{X = (x, y, z) : y \leq 0\}.$$

Note that P_4 is symmetric to P_2 and P_5 is symmetric to P_3 with respect to the plane

$$\{X = (x, y, z) : y = 0\}.$$

Strictly speaking, $U = P_1 \cup \dots \cup P_5$ is a cover rather than a partition (P_i and P_j may have common boundary points), but this is not that important.

In subsequent sections, we determine the diameters of the sets $P_1, \dots, P_5$ as functions of all parameters on which they depend; at the end, we perform an optimization, which yields the bound claimed in Theorem 1.

5. DETERMINATION OF THE DIAMETERS OF PARTS FOR $d \geq 0.592$

5.1. Simplest Observations

Note at once that the diameter of P_1 is equal to that of the circle

$$S \cap \{X = (x, y, z) : x = d - \varepsilon\};$$

i.e.,

$$\operatorname{diam} P_1 = 2\sqrt{d^2 - (d - \varepsilon)^2} = 2\sqrt{2d\varepsilon - \varepsilon^2}.$$

The sets P_2 and P_3 are compact convex bodies with piecewise smooth boundaries consisting of (open) domains in the plane and the sphere (we denote them by $\partial^2 P_i$ for $i = 2, 3$), open arcs of circles and intervals (we denote them by $\partial^1 P_i$ for $i = 2, 3$), and points (we denote them by $\partial^0 P_i$ for $i = 2, 3$). It would be easy to find the values of $\operatorname{diam} P_2$ and $\operatorname{diam} P_3$ if they were attained on $\partial^0 P_2$ and $\partial^0 P_3$, respectively. Fortunately, this is so, which we prove in the next two sections.

For convenience, we assume that $\varepsilon < 0.25$ and $\delta < 0.025$ (we can do this, because the required optimum is attained under these constraints). In this case, the cover and its partition look in principle as shown in Fig. 2.

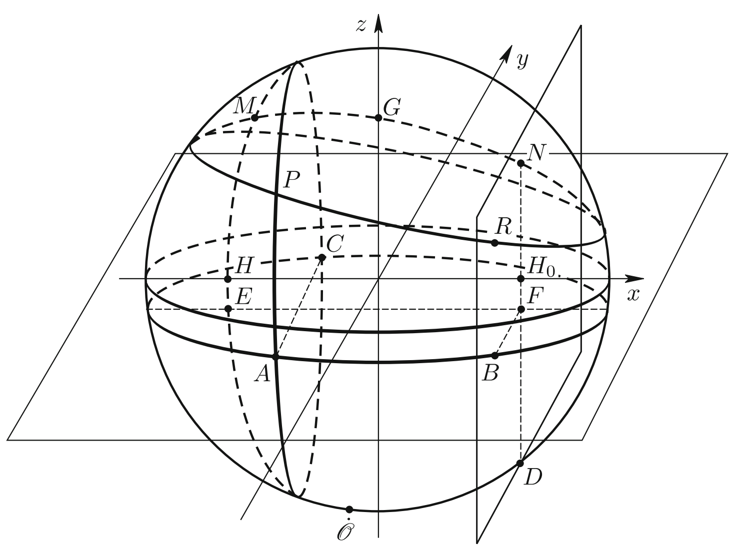

Fig. 2.

5.2. Reduction to the One-Dimensional Boundary Components

The following lemma is valid.

Lemma 2. *The diameter of P_2 is attained on $\partial^1 P_2 \cup \partial^0 P_2$. A similar assertion is valid for P_3.*

We prove Lemma 2 only for P_2; the case of P_3 is similar.

Proof of Lemma 2. Let $X, Y \in P_2$ be points for which $\rho(X, Y) = \operatorname{diam} P_2$. Such points exist, because P_2 is closed; obviously, $X, Y \in \partial P_2$. Suppose that the point Y belongs to the two-dimensional component of the boundary, i.e., $Y \in \partial^2 P_2$. There are two possible cases.

Case 1. In this case, Y belongs to a domain in the plane. Let us denote this domain by T. Take any points $A_1, A_2, A_3 \in T$ such that the triangle with vertices A_1, A_2, and A_3 contains Y. It is well known that the diameter of the tetrahedron $XA_1A_2A_3 \subset P_2$ is attained at a pair of its vertices. Thus,

$$\operatorname{diam} P_2 = \rho(X, Y) < \operatorname{diam} XA_1A_2A_3 \leq \operatorname{diam} P_2,$$

which is a contradiction.

Case 2. In this case, Y belongs to a spherical domain. Let us denote this domain by T. Then either $T \subset S$, $T \subset S_1$, or $T \subset S_2$. We write $T \subset S'$, meaning that S' is one of the spheres S, S_1, and S_2. Accordingly, we denote the center of the sphere S' by O'.

First, let us prove that $\overrightarrow{YX} \perp \pi_Y$, where π_Y is the tangent plane to S' at the point Y. Suppose that, on the contrary, there exists a vector $\overrightarrow{v} \in \pi_Y$ for which the inner product $(\overrightarrow{v}, \overrightarrow{YX})$ is negative. Consider the circle $W \subset S'$ with tangent vector $\overrightarrow{v}$ passing through Y. If a point $Z \in W$ is sufficiently close to Y and the direction of the vector $\overrightarrow{YZ}$ coincides with that of $\overrightarrow{v}$, then $Z \in T$, because T is open, and $(\overrightarrow{YX}, \overrightarrow{YZ}) < 0$, because the inner product is continuous. Therefore, $\rho(Z, X) > \rho(Y, X)$. But $Z, X \in P_2$, whence

$$\operatorname{diam} P_2 \geq \rho(Z, X) > \rho(Y, X) = \operatorname{diam} P_2.$$

We have arrived at a contradiction.

Thus, under the assumptions $\rho(X, Y) = \operatorname{diam} P_2$ and $Y \in T \subset S'$, we have $\overrightarrow{YX} \perp \pi_Y$. In other words, X belongs to the straight line YO'. Suppose that X is closer to Y than the center of the sphere. Consider the ball B of radius $\rho(X, Y)$ centered at X. Obviously, B is entirely contained in the ball D' bounded by the sphere S'. Hence, for any point Z in a sufficiently small neighborhood of Y on the sphere, we have simultaneously $Z \in T$ and $\rho(Z, X) > \rho(Y, X)$. We again obtain a contradiction.

The only remaining possibility is that the point X is "beyond" the center O' (or coincides with it). But this is absurd, because the centers of the spheres S_1 and S_2 do not belong to the part P_2 (because $\varepsilon > 0$, δ is small, and d is large), while the center of the sphere S belongs to the *plane component* of the boundary of this part ($\delta > 0$); thus, even if X coincides with the center of S, we still have a contradiction, as in Case 1.

In both Cases 1 and 2, we arrive at a contradiction to the initial assumption $Y \in \partial^2 P_2$, which proves the lemma. □

5.3. Reduction to the Zero-Dimensional Boundary Components

The following assertion is valid.

Lemma 3. *The diameter of P_2 is attained on $\partial^0 P_2$, and a similar assertion is valid for P_3.*

We prove Lemma 3 only for P_2. The case of P_3 is similar.

Proof of Lemma 3. First, note that if the intersection of two spheres is nonempty, then this is either a circle or a point, and the same is true for a sphere and a plane. We denote the (open) arc joining points X and Y on the sphere by $\widetilde{XY}$. Of course, such an arc is far from being unique; however, it is always clear which arc is considered from the context.

We know from Lemma 2 that, if $X, Y \in P_2$ and $\rho(X, Y) = \operatorname{diam} P_2$, then both points X and Y belong to the union $\partial^1 P_2 \cup \partial^0 P_2$. Suppose that, contrary to the assertion of Lemma 3, the point Y belongs to the one-dimensional component of the boundary, i.e., $Y \in \partial^1 P_2$.

This assumption means that Y is either inside one of the nine arcs

$$\widetilde{AB},\ \widetilde{AE},\ \widetilde{ME},\ \widetilde{MN},\ \widetilde{RN},\ \widetilde{RB},\qquad \widetilde{PM},\ \widetilde{PA},\ \widetilde{PR} \tag{2}$$

or inside one of the three intervals BF, NF, and EF (see Fig. 2). The first six arcs are intersections of planes with spheres, while the last three arcs are obtained without employing planes. Thus, three fundamentally different cases arise.

Case 1. If either $Y \in BF$, $Y \in NF$, or $Y \in EF$, then, applying the argument used to handle Case 1 in the proof of Lemma 2, we obtain a contradiction to the maximality of $\rho(X, Y)$ on the set P_2.

Case 2. Now, suppose that Y belongs to one of the first six arcs among those listed in (2). First, consider the situation in which $Y \in \widetilde{AB}$.

Clearly, the point Y is one of the farthest points from X in the set P_2. In other words, at Y, the function $\rho(X, Y')$ attains its maximum over all $Y' \in P_2$.

Let P_2' denote the orthogonal projection of the set P_2 on the plane

$$\Pi = \{Z = (x, y, z) : z = -\delta\} \supset \widetilde{AB},$$

and let X' be the image of X under the projection. We set $P_2'' = P_2 \cap \Pi \subset P_2'$. As mentioned above, the function $\rho(X, Y')$ attains its maximum over $Y' \in P_2$ at Y; therefore, the function $\rho(X', Y')$ also attains its maximum over $Y' \in P_2''$ at Y.

Let S' be the circle in Π containing $\widetilde{AB}$. We denote its center by O' and the tangent to S' at the point Y by π_Y ($\pi_Y \subset \Pi$). In principle, there are two possibilities: either the points X' and O' are on different sides of the line π_Y, or they belong to the same half-plane. The first possibility drops out at once by virtue of the maximality of $\rho(X', Y)$ mentioned above. Consider the second possibility. The same argument as at the beginning of the proof of Case 2 in Lemma 2 (see Sec. 5.2), together with the maximality of $\rho(X', Y)$, implies $\overrightarrow{YX'} \perp \pi_Y$. Consequently, X' belongs to the line YO'.

Again applying the argument from the proof of Case 2 in Lemma 2, we see that X' cannot be closer to Y than O'. Therefore, the only possible case is $X' = O'$. In this case,

$$X = G \qquad \text{and} \qquad \rho(X, Y) = \rho(G, Y) = \rho(G, B)$$

(see Fig. 2). It is easy to show that if G' is any point of the arc $\widetilde{GM}$ from a sufficiently small neighborhood of G, then $\rho(G', B) > \rho(G, B)$. This contradicts the maximality of $\rho(X, Y)$; therefore, $Y \notin \widetilde{AB}$.

From now on, we assume that $Y \in \widetilde{RB}$. Applying the same trick of projecting P_2 onto the plane Π, which is defined this time as

$$\{Z = (x, y, z) : x = d - \varepsilon\} \supset \widetilde{RB},$$

we conclude that

$$X = H \qquad \text{and} \qquad \rho(X, Y) = \rho(H, Y) = \rho(H, R)$$

(see Fig. 2). Shifting along the arc $\widetilde{HE}$ toward E yields points H' at which $\rho(H', R) > \rho(H, R)$. This again contradicts the maximality of $\rho(X, Y)$. Thus, $Y \notin \widetilde{RB}$.

Now, the general scheme of the proof that $Y \notin \widetilde{I}$ for all

$$\widetilde{I} \in \{\widetilde{AE}, \widetilde{ME}, \widetilde{MN}, \widetilde{RN}\}$$

is quite clear. We denote the plane containing $\widetilde{I}$ by Π, consider the orthogonal projection P_2' of the set P_2 on Π, and let $P_2'' = P_2 \cap \Pi$. Next, we consider the image X' of the point X under the projection, the circle S' in Π containing $\widetilde{I}$, and its center O' and show that X' must belong to the line YO' and cannot be closer to Y than O'. Unlike in the cases $\widetilde{I} \in \{\widetilde{AB}, \widetilde{RB}\}$, in which X' may coincide with O', we have $O' \notin P_2$ in each of the four cases under consideration, which leads immediately to a contradiction.

This completes the consideration of Case 2.

Case 3. Finally, suppose that Y belongs to one of the last three arcs among those listed in (2). First, consider the case $Y \in \widetilde{PA}$.

In this case, the argument used at the end of the proof of the preceding case readily applies. Of course, $\widetilde{PA}$ is contained in none of the planes generating P_2, but it is contained in the plane

$$\Pi = \{Z = (x, y, z) : x = \text{const}\}.$$

Projecting the set P_2 on the plane Π and considering its intersection with this plane, we prove in a standard way that the point X at which the maximum of $\rho(X, Y)$ is attained must coincide with H or H_0. But this is impossible, because, as we showed in the proof of the first two cases, $X \notin \widetilde{ME}$ and $X \notin \widetilde{NF}$.

Thus, the only remaining possibility for the points X and Y is to belong to the arcs $\widetilde{PM}$ and $\widetilde{PR}$.

In principle, it is obvious from practical considerations that the diameter of P_2 cannot be realized at points of the arcs $\widetilde{PM}$ and $\widetilde{PR}$, but this can also be proved by using the standard method of projection on a plane. In this case, it is more convenient to deal with the arc $\widetilde{PM}$. The plane of this arc is perpendicular to the line joining the centers of the spheres S_1 and S_2. Further considerations are standard.

This completes the proof of the lemma. □

5.4. The Final Localization of Diameters

Lemma 4. *The diameter of P_3 is realized by one of the intervals BE, AD, and BC.*

Proof of Lemma 4. By virtue of the lemmas proved in the preceding sections, the diameter can be realized only by one of the $C_6^2 = 15$ intervals

$$AE,\ AB,\ AF,\ EF,\ BF,\ AC,\ BD,\ CD,\ CE,\ CF,\ DF,\ DE,\ BE,\ AD,\ BC$$

(here and elsewhere, we use the notation $C_n^k = \binom{n}{k}$).

The first group of intervals drops out at once. For example, AE does not realize the diameter, because for any point $A' \in \widetilde{AB}$ sufficiently close to A, we have $\rho(A', E) > \rho(A, E)$. In the other cases, the proof is similar.

The case of the interval DE is more involved. Consider the set Π of points in space equidistant from A and E. This is the plane passing through the midpoint of the arc $\widetilde{AE}$ and perpendicular to this arc. Since the points A and E belong to the sphere S_1, it follows that the center of this sphere is equidistant from these points and, therefore, belongs to the plane Π. At the same time, obviously, the abscissa of D is less than that of the center of S_1. Therefore, A and D are on different sides of Π, and E and D belong to the same half-space. This proves the inequality $\rho(D, E) < \rho(D, A)$ and the lemma. □

Remark. Our ultimate interest is in the maximal diameter of $P_1, \dots, P_5$. However, clearly, the interval BE is shorter than the diameter of P_2. This means that the interval BE can be eliminated from consideration, although it belongs to the set of intervals that are candidates for realizing the diameter P_3 by virtue of Lemma 4.

Thus, in what follows, we measure only AD and BC.

Lemma 5. *The diameter of P_2 is realized by one of the intervals PF, RE, BM, and AN.*

Proof of Lemma 5. We argue as in the proof of the preceding lemma. There are $C_8^2 = 28$ pairs of points at which the diameter may be attained. Twenty three of them drop out from obvious considerations, four pairs are those specified in the statement of the lemma, and one "doubtful" pair is EN. Consider the same plane Π as in the proof of Lemma 4. It is easy to see that A and N are on different sides of this plane, while E and N belong to the same half-space. Therefore, $\rho(E, N) < \rho(A, N)$, which proves the lemma. □

5.5. Summary of Calculations

It remains to calculate

$$\rho(A, D), \quad \rho(B, C), \quad \rho(P, F), \quad \rho(R, E), \quad \rho(B, M), \quad \rho(A, N).$$

Certainly, these values depend on $U \in \mathscr{U}$ (that is, eventually, on d and $\mathscr{O}$), on ε, and on δ. Note however that, for fixed d, the position of the point $\mathscr{O}$ does not affect $\rho(A, D)$ and $\rho(B, C)$. At the same time, obviously, the values of $\rho(P, F)$ and $\rho(B, M)$, are maximal for $\mathscr{O} = C$, while, on the contrary, $\rho(A, N)$ and $\rho(R, E)$ are maximal for $\mathscr{O} = (0, 0, -d)$. Thus, we determine the "extreme" positions of M, P and R, N and calculate the required distances only for these positions. Thereby, we obviate the necessity for an exhaustive search over $\mathscr{O}$.

It is easy to show that, under these reservations, we have

$$A = \left(d - \frac{1}{2d}, -\sqrt{1 - \frac{1}{4d^2} - \delta^2}, -\delta\right), \qquad B = (d - \varepsilon, -\sqrt{2d\varepsilon - \varepsilon^2 - \delta^2}, -\delta),$$

$$C = \left(d - \frac{1}{2d}, 0, -\sqrt{1 - \frac{1}{4d^2}}\right), \qquad D = (d - \varepsilon, 0, -\sqrt{2d\varepsilon - \varepsilon^2}),$$

$$M = \left(d - \frac{1 + \sqrt{3(4d^2 - 1)}}{4d}, 0, -\frac{\sqrt{4d^2 - 1} - \sqrt{3}}{4d}\right),$$

$$N = (d - \varepsilon, 0, -d + \sqrt{1 - (d - \varepsilon^2)}),$$

$$E = (d - \sqrt{1 - \delta^2}, 0, -\delta), \qquad F = (d - \varepsilon, 0, -\delta),$$

$$P = \left(d - \frac{1}{2d}, -\sqrt{\frac{3 - 1/d^2}{4 - 1/d^2}}, \frac{1/2d^2 - 1}{\sqrt{4 - 1/d^2}}\right),$$

$$R = \left(d - \varepsilon, -\sqrt{1 - \frac{1}{4d^2} - (d - \varepsilon)^2}, \frac{1}{2d} - d\right).$$

Let us find the lengths of the intervals of interest for us:

$$\rho^2(A, D) = 1 + 2d\varepsilon - \frac{\varepsilon}{d} - 2\delta\sqrt{2d\varepsilon - \varepsilon^2},$$

$$\rho^2(B, C) = 1 + 2d\varepsilon - \frac{\varepsilon}{d} - 2\delta\sqrt{1 - \frac{1}{4d^2}},$$

$$\rho^2(A, N) = 2 + 2d\varepsilon - 2d\delta - \frac{\varepsilon}{d} - 2(d - \delta)\sqrt{1 - (d - \varepsilon)^2},$$

$$\rho^2(B, M) = 1 + 2d\varepsilon - \frac{\varepsilon}{2d}(1 + \sqrt{3(4d^2 - 1)}) - \frac{\delta}{2d}(\sqrt{4d^2 - 1} - \sqrt{3}),$$

$$\rho^2(P, F) = 1 + \varepsilon^2 + \delta^2 - \frac{\varepsilon}{d} + \frac{1 - 2d^2}{d\sqrt{4d^2 - 1}}\delta,$$

$$\rho^2(R, E) = 1 - \delta\left(2d - \frac{1}{d}\right) + 2\varepsilon(d - \sqrt{1 - \delta^2}).$$

5.6. The Choice of Parameters and Completion of Calculations

For each $d \geq 0.592$, we set

$$\delta = 0.0228 - \frac{3}{5}\left(\sqrt{\frac{3}{8}} - d\right), \qquad \varepsilon = 0.2212 + \frac{2}{5}\left(\sqrt{\frac{3}{8}} - d\right).$$

It is easy to see that, in this case, we have

$$0.592 \leq d \leq 0.613, \qquad 0.22 \leq \varepsilon \leq 0.23, \qquad 0.01 \leq \delta \leq 0.023. \tag{3}$$

In particular, ε and δ satisfy the inequalities from Sec. 5.1.

Now, let us show that, for all admissible values of d, (3) implies already

$$\rho(A, N) < 0.94, \qquad \rho(R, E) < 0.94, \qquad \rho(P, F) < 0.94.$$

Indeed, we have

$$\rho^2(A, N) \leq 2 + 2 \cdot 0.613 \cdot 0.23 - 2 \cdot 0.592 \cdot 0.01 - \frac{0.22}{0.613}$$
$$- 2 \cdot (0.592 - 0.023) \cdot \sqrt{1 - (0.613 - 0.22)^2} < 0.87,$$

$$\rho(A, N) < \sqrt{0.87} < 0.94,$$

$$\rho^2(R, E) \leq 1 + 0.023 \cdot \left(\frac{1}{0.592} - 2 \cdot 0.592\right) + 2 \cdot 0.23 \cdot \left(0.613 - \sqrt{1 - (0.023)^2}\right) < 0.84,$$

$$\rho(R, E) < \sqrt{0.84} < 0.94,$$

$$\rho^2(P,F) \le 1 + (0.23)^2 + (0.023)^2 - \frac{0.22}{0.613} + 0.023 \cdot \frac{1 - 2\cdot(0.592)^2}{0.592\sqrt{4\cdot(0.592)^2 - 1}} < 0.8,$$

$$\rho(P,F) < \sqrt{0.8} < 0.94.$$

Thus, it only remains to determine $\rho^2(A,D)$, $\rho^2(B,C)$, and $\rho^2(B,M)$. Let us prove the inequality

$$\rho^2(A,D) > \rho^2(B,C).$$

Indeed, we have

$$\frac{\rho^2(A,D) - \rho^2(B,C)}{2\delta} = \sqrt{1 - \frac{1}{4d^2}} - \sqrt{2d\varepsilon - \varepsilon^2} > \sqrt{1 - \frac{1}{4\cdot(0.592)^2}} - \sqrt{2\cdot 0.613 \cdot 0.23} > 0.$$

Thus, we must consider the three quantities

$$f_1 = \rho^2(A,D), \qquad f_2 = \rho^2(B,M), \qquad f_3 = \operatorname{diam} P_1 = 2\sqrt{2d\varepsilon - \varepsilon^2}.$$

By virtue of the parameterization introduced at the beginning of this section, we have $f_i = f_i(d)$ for $i = 1,2,3$. Let us show that the functions f_i increase with respect to $d \in [0.592, r]$. To this end, we simply estimate their derivatives.

We have

$$((f_3/2)^2)' = 2\varepsilon - \frac{4}{5}d + \frac{4}{5}\varepsilon \ge 0.44 - \frac{4}{5}\cdot 0.613 + \frac{4}{5}\cdot 0.23 > 0,$$

$$\begin{aligned} f_1' &= 2\varepsilon - \frac{4}{5}d + \frac{(2/5)d + \varepsilon}{d^2} - \delta\frac{(2\varepsilon - (4/5)d + (4/5)\varepsilon)}{\sqrt{2d\varepsilon - \varepsilon^2}} - \frac{6}{5}\sqrt{2d\varepsilon - \varepsilon^2} \\ &\ge 2\cdot 0.22 - \frac{4}{5}\cdot 0.613 + \frac{2}{5\cdot 0.613} + \frac{0.22}{(0.613)^2} - \frac{(14/5)\cdot 0.023\cdot 0.23}{\sqrt{2\cdot 0.592\cdot 0.22 - (0.23)^2}} \\ &\quad + \frac{(4/5)\cdot 0.592\cdot 0.01}{\sqrt{2\cdot 0.613\cdot 0.23 - (0.22)^2}} - \frac{6}{5}\sqrt{2\cdot 0.613\cdot 0.23 - (0.22)^2} > 0, \end{aligned}$$

$$\begin{aligned} f_2' &= 2\varepsilon - \frac{4}{5}d - \frac{\varepsilon}{2d}\frac{12d}{\sqrt{3(4d^2-1)}} + (1 + \sqrt{3(4d^2-1)})\frac{(4/5)d + 2\varepsilon}{4d^2} \\ &\quad - \frac{\delta}{2d}\frac{4d}{\sqrt{4d^2-1}} + (\sqrt{3} - \sqrt{4d^2-1})\frac{(6/5)d - 2\delta}{4d^2} \\ &\ge 2\cdot 0.22 - \frac{4}{5}\cdot 0.613 - \frac{0.23\cdot 6}{\sqrt{3(4(0.592)^2-1)}} \\ &\quad + (1 + \sqrt{3(4(0.592)^2-1)})\frac{(4/5)\cdot 0.592 + 2\cdot 0.22}{4(0.613)^2} - \frac{2\cdot 0.023}{\sqrt{4(0.592)^2-1}} \\ &\quad + (\sqrt{3} - \sqrt{4(0.613)^2-1})\frac{(6/5)\cdot 0.592 - 2\cdot 0.023}{4(0.613)^2} > 0. \end{aligned}$$

Therefore, under the above constraints and parameterization, the quantities f_1, f_2, and f_3 attain their maxima at

$$d = \sqrt{\frac{3}{8}}, \qquad \varepsilon = 0.2212, \qquad \delta = 0.0228.$$

Substituting these parameter values in the expressions for f_1, f_2, and f_3, we obtain

$$\rho(A,D) = 0.94244\dots < 0.9425, \qquad \rho(B,M) = 0.94243\dots < 0.9425,$$
$$\operatorname{diam} P_1 = 0.9423\dots < 0.9425.$$

This completes the consideration of the case $d \geq 0.592$. Note that we have not made an effort to achieve absolute accuracy in the choice of the parameters, because improving the bound at the fourth or fifth decimal place is of little interest.

6. FINDING THE DIAMETERS OF PARTS FOR $d < 0.592$

We shall estimate the diameters of the parts which we are interested in as if there were no sphere S_2 in the cover and in the figure. In essence, we obtain something similar to the Lassak cover. As δ tends to zero, the difference between P_2 and P_3 becomes infinitesimally small. Thus, we can assume that the parts $P_2, \dots, P_5$ are almost the same and their diameters are virtually indistinguishable from the maximum of $\rho(A, D)$ and $\rho(B, C)$. We have (for $\delta \sim 0$)

$$\rho^2(A, D) \sim \rho^2(B, C) \sim 1 + 2d\varepsilon - \frac{\varepsilon}{d},$$
$$\operatorname{diam} P_1 = 2\sqrt{2d\varepsilon - \varepsilon^2}.$$

Obviously, for any fixed ε, the quantities after the $\sim$ and $=$ signs increase with d. Let $\varepsilon = 0.232$. Then simple calculations show that, even for $d = 0.592$, the values of the first two quantities are at most $(0.9399)^2$, and the value of the last is at most 0.9399. Therefore, for sufficiently small δ, we have

$$\rho(A, D) \leq 0.94, \qquad \rho(B, C) \leq 0.94,$$
$$\operatorname{diam} P_1 \leq 0.94.$$

This completes the analysis of the case $d < 0.592$ and the proof of the theorem.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research (grant no. 06-01-00383), the program "Leading Scientific Schools" (grant no. NSh-691.2008.1), the Program for Support of Young Candidates of Sciences (grant no. MD-5414.2008.1), and the "Dynasty" foundation.

REFERENCES

1. A. M. Raigorodskii, "The Borsuk problem and the chromatic numbers of some metric spaces," Uspekhi Mat. Nauk **56** (1), 107–146 (2001) [Russian Math. Surveys **56** (1), 103–139 (2001)].
2. A. M. Raigorodskii, "Around the Borsuk conjecture," *Contemporary Mathematics: Fundamental Directions*, Vol. 23: *Geometry and mechanics* (Ross. Univ. Druzhby Narodov, Moscow, 2007), Vol. 23, pp. 147–164 [J. Math. Sci. (New York) *154* (4), 604–623 (2008)].
3. A. M. Raigorodskii, "Three lectures on the Borsuk partition problem," in *London Math. Soc. Lecture Note Ser.*, Vol. 347: *Surveys in Contemporary Mathematics* (Cambridge Univ. Press, Cambridge, 2008), pp. 202–247.
4. A. M. Raigorodskii, "The Borsuk partition problem: The seventieth anniversary," Math. Intelligencer **26** (3), 4–12 (2004).
5. A. M. Raigorodskii, *The Borsuk Problem* (MTsNMO, Moscow, 2006) [in Russian].
6. A. M. Raigorodskii, *A Linear-Algebraic Method in Combinatorics* (MTsNMO, Moscow, 2007) [in Russian].
7. V. Boltyanski, H. Martini, and P. S. Soltan, *Excursions into Combinatorial Geometry*, in *Universitext* (Springer-Verlag, Berlin, 1997).
8. V. G. Boltyanskii and I. Ts. Gokhberg, *Theorems and Problems in Combinatorial Geometry* (Nauka, Moscow, 1965) [in Russian].
9. H. Lenz, "Zerlegung ebener Bereiche in konvexe Zellen von möglichst kleinem Durchmesser," Jahresber. Deutsch. Math.-Verein. **58**, 87–97 (1956).
10. H. Lenz, "Über die Bedeckung ebener Punktmengen durch solche kleineren Durchmesser," Arch. Math. (Basel) **7**, 34–40 (1956).
11. V. P. Filimonov, "On covers of plane sets," Mat. Sb. (in press).
12. D. Gale, "On inscribing n-dimensional sets in a regular n-simplex," Proc. Amer. Math. Soc. **4** (2), 222–225 (1953).
13. B. Grünbaum, "A simple proof of Borsuk's conjecture in three dimensions," Proc. Cambridge Philos. Soc. **53**, 776–778 (1957).

14. A. Heppes, “On the partitioning of three-dimensional point-sets into sets of smaller diameter,” Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **7**, 413–416 (1957).
15. V. V. Makeev, “On affine images of a rhombo-dodecahedron circumscribed about a three-dimensional convex body,” in *Zapiski Nauchn. Sem. St. Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, Vol. 246: *Geometry and topology. 2* (POMI, St. Peterburg., 1997), pp. 191–195 [J. Math. Sci. (New York) **100** (3), 2307–2309 (2000)].
16. M. Lassak, “An estimate concerning Borsuk partition problem,” Bull. Acad. Polon. Sci. Sér. Sci. Math. **30** (9–10), 449–451 (1982).
17. L. Danzer, B. Grünbaum, and V. Klee, *Helly's Theorem and Its Relatives* (Mir, Moscow, 1968) [*Proc. Sympos. Pure Math.* (Amer. Math. Soc., Providence, RI, 1963), Vol. 7, pp. 101–180].
18. A. M. Raigorodskii and Yu. A. Kalnishkan, “On the Borsuk problem in $\mathbb{R}^3$,” Mat. Zametki **74** (1), 149–151 (2003) [Math. Notes **74** (1–2), 144–146 (2003)].
19. B. Weissbach, “Polyhedral covers,” in *Colloq. Math. Soc. János Bolyai*, Vol. 48: *Intuitive Geometry*, Siófok, 1985 (North-Holland, Amsterdam, 1987), pp. 639–646.
20. B. B. Makeev, “Universal coverings and projections of bodies constant width,” Ukrain. Geom. Sb., No. 32, 84–88 (1989) [J. Soviet Math. **59** (2), 750–752 (1992)].