On the chromatic number of small-dimensional Euclidean spaces

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Chromatic number

- $$F = F_m(x)\colon \mathbb{R}^n \xrightarrow{F} \{1, \ldots, m\}, \quad (F(x_0) = F(x)) \Rightarrow |x - x_0| \neq 1$$
- $$\chi(\mathbb{R}^n) = \min\{m \in \mathbb{N} : \exists F_m\}.$$
- $4 \leq \chi(\mathbb{R}^2) \leq 7$
- $(1,239 + o(1))^n \leq \chi(\mathbb{R}^n) \leq (3 + o(1))^n$ – Raigorodskii; Larman, Rogers

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Results in low dimensions

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dim	1	2	3	4	5	6
$\chi \geq$	2	4	6	7	9	11

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dim	7	8	9	10	11	12
$\chi \geq$	15	16	16	19	20	24

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Result in $\mathbb{R}^9$

Theorem 1

The inequality holds $\chi(\mathbb{R}^9) \geq 21$.

- We call a graph $W = (V, E)$ *unit-distance* in metric space Γ, if $V \subset (\Gamma, \rho)$, and $\forall (x_1, x_2) \in E \; \rho(x_1, x_2) = 1$, where ρ is metric in Γ.
- Theorem 2 (Raiskii)
  Take G – unit-distance graph on the sphere $S^{n-2} \subset \mathbb{R}^{n-1}$ with radius $r < \frac{\sqrt{15}}{4}$. Then we can construct a unit-distance graph in $\mathbb{R}^n$ with chromatic number at least $\chi(G) + 2$.
- Theorem 3
  Take G – unit-distance graph on the sphere $S^{n-2} \subset \mathbb{R}^{n-1}$, $n \geq 3$, with radius r_s, $1/2 \leq r_s \leq \sqrt{\frac{1+\sqrt{3}}{2+\sqrt{3}}} \approx 0.856, \quad r_s \neq \sqrt{2/3}$. Then we can construct a unit-distance graph in $\mathbb{R}^{n+1}$ with chromatic number at least $\chi(G) + 4$.

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- **Corollary** (from theorems 1,2,3)
  The inequalities hold $\chi(\mathbb{R}^{10}) \geq 23$, $\chi(\mathbb{R}^{11}) \geq 25$.
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Sketch of proof of theorem 1. Construction

1. we consider a graph G of (10,5,3)-vectors – $G = (V, E)$ with $V = \{v = (v_1, \ldots, v_{10}), v_i \in \{0,1\}, v_1 + \ldots + v_{10} = 5\}$, $E = \{\{u, v\} \in V \times V,\ (u, v) = u_1 v_1 + \ldots + u_{10} v_{10} = 3\}$.
2. $\alpha(G)$ is the maximal power of subset of the set V such that each pair of vertices from the subset is not connected by edge ($\alpha(G)$ – *independence number*).
3. $|V| = 252, \quad \chi(G) \geq |V|/\alpha(G)$.
4. **Theorem 4** $\alpha(G) = 12$.
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Outline of proof of theorem 4

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1	1	1	1	1	0	0	0	0	0
0	0	0	0	1	1	1	1	1	0
1	1	1	1	0	C_4^1				0
C_4^1				0	1	1	1	1	0
0	0	0	0	0	1	1	1	1	1
1	1	1	1	0	0	0	0	0	1

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3. We enumerate possibilities of how can independent set look, using some symmetry of set V, starting from two vectors from lemma 1.

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- **Lemma 2**
  $\forall r > 1/2 \ \ \forall \varepsilon > 0 \ \ \exists r_0 < r, |r - r_0| < \varepsilon$, so that every circle S_{r_0} with radius r_0 contains cycle of odd length with unit edges.
- **Corollary**
  For arbitrary coloring, for all $r > 1/2$ and for arbitrary fixed color k sphere $S_r^2 \subset \mathbb{R}^3$ with radius r contains a unit edge connecting vertices, both painted in color that differs from k.
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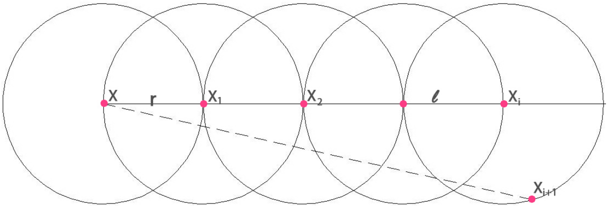

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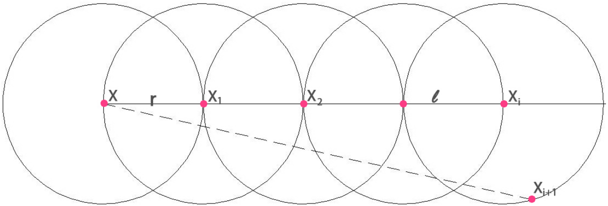

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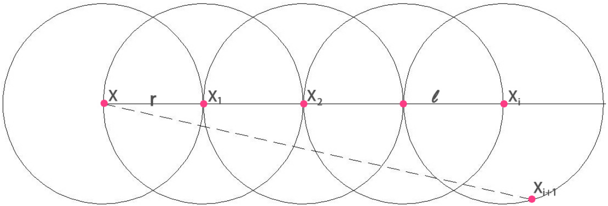
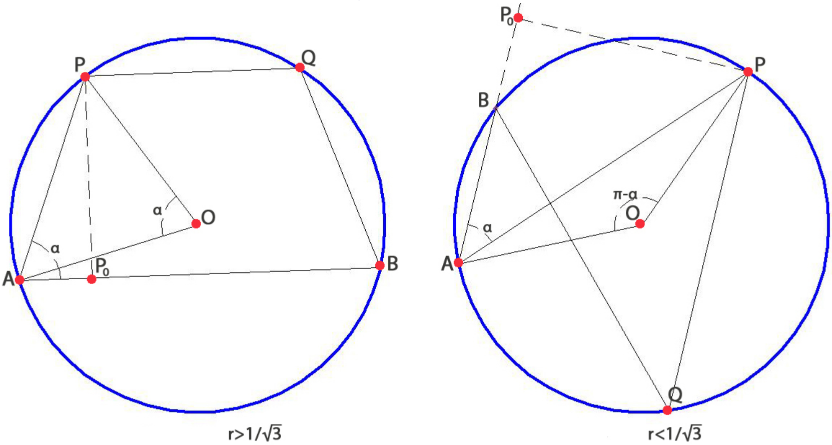


End of proof



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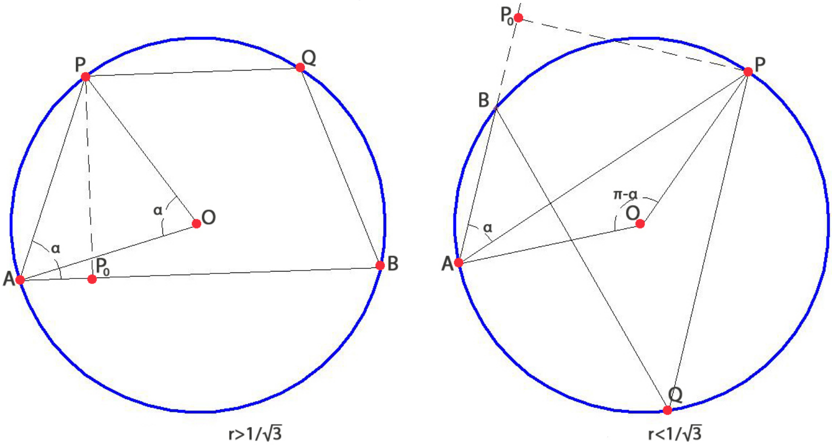

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Thank You