On dividing three-dimensional sets into five parts of smaller diameter

Kupavskiy A.B. Raigorodskiy A.M.

Department of Mechanics and Mathematics
Moscow State University
Moscow, Russia

August 12, 2008, Keszthely

Borsuk's problem

- $f(\Omega) = \min\{ f : \Omega = \Omega_1 \cup \ldots \cup \Omega_f, \operatorname{diam} \Omega_i < \operatorname{diam} \Omega\}$
- $f(n) = \max_{\Omega,\ \Omega \subset \mathbb{R}^n} f(\Omega), \quad \operatorname{diam} \Omega < \infty$
- $f(n) = n + 1$ K. Borsuk's conjecture, 1933
- $f(n) = n + 1$ for $n \leq 3$
  $f(n) > n + 1$ for $n \geq 298$
- $(1.2255\ldots + o(1))^{\sqrt{n}} \leq f(n) \leq (1.224\ldots + o(1))^n$

Borsuk's problem

- $f(\Omega) = \min\{\ f : \Omega = \Omega_1 \cup \ldots \cup \Omega_f, \operatorname{diam} \Omega_i < \operatorname{diam} \Omega\}$
- $f(n) = \max\limits_{\Omega,\ \Omega \subset \mathbb{R}^n} f(\Omega), \quad \operatorname{diam} \Omega < \infty$
- $f(n) = n + 1$ K. Borsuk's conjecture, 1933
- $f(n) = n + 1$ for $n \leq 3$
  $f(n) > n + 1$ for $n \geq 298$
- $(1.2255\ldots + o(1))^{\sqrt{n}} \leq f(n) \leq (1.224\ldots + o(1))^n$

Borsuk's problem

- $f(\Omega) = \min\{\ f : \Omega = \Omega_1 \cup \ldots \cup \Omega_f, \operatorname{diam} \Omega_i < \operatorname{diam} \Omega\}$
- $f(n) = \max\limits_{\Omega,\ \Omega \subset \mathbb{R}^n} f(\Omega), \quad \operatorname{diam} \Omega < \infty$
- $f(n) = n + 1$ K. Borsuk's conjecture, 1933
- $f(n) = n + 1$ for $n \leq 3$
  $f(n) > n + 1$ for $n \geq 298$
- $(1.2255\ldots + o(1))^{\sqrt{n}} \leq f(n) \leq (1.224\ldots + o(1))^n$

Borsuk's problem

- $f(\Omega) = \min\{\ f : \Omega = \Omega_1 \cup \ldots \cup \Omega_f, \operatorname{diam}\Omega_i < \operatorname{diam}\Omega\}$
- $f(n) = \max\limits_{\Omega,\ \Omega \subset \mathbb{R}^n} f(\Omega), \quad \operatorname{diam}\Omega < \infty$
- $f(n) = n + 1$ K. Borsuk's conjecture, 1933
- $f(n) = n + 1$ for $n \leq 3$
  $f(n) > n + 1$ for $n \geq 298$
- $(1.2255\ldots + o(1))^{\sqrt{n}} \leq f(n) \leq (1.224\ldots + o(1))^n$

Borsuk's problem

- $f(\Omega) = \min\{\ f : \Omega = \Omega_1 \cup \ldots \cup \Omega_f, \operatorname{diam} \Omega_i < \operatorname{diam} \Omega\}$
- $f(n) = \max\limits_{\Omega,\ \Omega \subset \mathbb{R}^n} f(\Omega), \quad \operatorname{diam} \Omega < \infty$
- $f(n) = n + 1$ K. Borsuk's conjecture, 1933
- $f(n) = n + 1$ for $n \leq 3$
  $f(n) > n + 1$ for $n \geq 298$
- $(1.2255\ldots + o(1))^{\sqrt{n}} \leq f(n) \leq (1.224\ldots + o(1))^n$

Definition of d_k^n

- $$\forall\ \Phi \subset \mathbb{R}^n$$
  $$d_k^n(\Phi) = \inf\{x \geq 0 :\ \Phi = \Phi_1 \cup \Phi_2 \cup \ldots \cup \Phi_k, \operatorname{diam} \Phi_i \leq x\}$$
- $$d_k^n = \sup_{\Phi,\ \operatorname{diam} \Phi = 1} d_k^n(\Phi).$$

Definition of d_k^n

- $$\forall\ \Phi \subset \mathbb{R}^n$$
  $$d_k^n(\Phi) = \inf\{x \geq 0 :\ \Phi = \Phi_1 \cup \Phi_2 \cup \ldots \cup \Phi_k, \operatorname{diam} \Phi_i \leq x\}$$
- $$d_k^n = \sup_{\Phi,\ \operatorname{diam} \Phi = 1} d_k^n(\Phi).$$

Results in $\mathbb{R}^2$

1. $d_2^1 = d_2^2 = 1$
2. $d_3^2 = \frac{\sqrt{3}}{2}$
3. $d_4^2 = \frac{1}{\sqrt{2}}$ – H. Lenz, 1956
4. $d_7^2 = \frac{1}{2}$ – H. Lenz, 1956
5. d_k^2 for many other cases – V. Filimonov, 2008

Results in $\mathbb{R}^2$

1. $d_2^1 = d_2^2 = 1$
2. $d_3^2 = \frac{\sqrt{3}}{2}$
3. $d_4^2 = \frac{1}{\sqrt{2}}$ – H. Lenz, 1956
4. $d_7^2 = \frac{1}{2}$ – H. Lenz, 1956
5. d_k^2 for many other cases – V. Filimonov, 2008

Results in $\mathbb{R}^2$

1. $d_2^1 = d_2^2 = 1$
2. $d_3^2 = \frac{\sqrt{3}}{2}$
3. $d_4^2 = \frac{1}{\sqrt{2}}$ – H. Lenz, 1956
4. $d_7^2 = \frac{1}{2}$ – H. Lenz, 1956
5. d_k^2 for many other cases – V. Filimonov, 2008

Results in $\mathbb{R}^2$

1. $d_2^1 = d_2^2 = 1$
2. $d_3^2 = \frac{\sqrt{3}}{2}$
3. $d_4^2 = \frac{1}{\sqrt{2}}$ – H. Lenz, 1956
4. $d_7^2 = \frac{1}{2}$ – H. Lenz, 1956
5. d_k^2 for many other cases – V. Filimonov, 2008

Results in $\mathbb{R}^2$

1. $d_2^1 = d_2^2 = 1$
2. $d_3^2 = \frac{\sqrt{3}}{2}$
3. $d_4^2 = \frac{1}{\sqrt{2}}$ – H. Lenz, 1956
4. $d_7^2 = \frac{1}{2}$ – H. Lenz, 1956
5. d_k^2 for many other cases – V. Filimonov, 2008

Some Results in $\mathbb{R}^3$

1. $d_1^3 = d_2^3 = d_3^3 = 1$
2. $d_4^3 \geq \sqrt{(3+\sqrt{3})/6} = 0.888\ldots$
3. $d_4^3 = \sqrt{(3+\sqrt{3})/6}$ – D. Gale conjecture, 1953
4. $d_4^3 \leq 0.9977$ – A. Heppes, 1957
5. $d_4^3 \leq 0.9887$ – B. Grünbaum, 1957
6. $d_4^3 \leq 0.98$ – V. Makeev, L. Evdokimov, 1997
7. $d_5^3 \leq \sqrt{(35+\sqrt{73})/48} = 0.9524\ldots$ – M. Lassak, 1982

Some Results in $\mathbb{R}^3$

1. $d_1^3 = d_2^3 = d_3^3 = 1$
2. $d_4^3 \geq \sqrt{(3+\sqrt{3})/6} = 0.888\ldots$
3. $d_4^3 = \sqrt{(3+\sqrt{3})/6}$ – D. Gale conjecture, 1953
4. $d_4^3 \leq 0.9977$ – A. Heppes, 1957
5. $d_4^3 \leq 0.9887$ – B. Grünbaum, 1957
6. $d_4^3 \leq 0.98$ – V. Makeev, L. Evdokimov, 1997
7. $d_5^3 \leq \sqrt{(35+\sqrt{73})/48} = 0.9524\ldots$ – M. Lassak, 1982

Some Results in $\mathbb{R}^3$

1. $d_1^3 = d_2^3 = d_3^3 = 1$
2. $d_4^3 \geq \sqrt{(3+\sqrt{3})/6} = 0.888\ldots$
3. $d_4^3 = \sqrt{(3+\sqrt{3})/6}$ – D. Gale conjecture, 1953
4. $d_4^3 \leq 0.9977$ – A. Heppes, 1957
5. $d_4^3 \leq 0.9887$ – B. Grünbaum, 1957
6. $d_4^3 \leq 0.98$ – V. Makeev, L. Evdokimov, 1997
7. $d_5^3 \leq \sqrt{(35+\sqrt{73})/48} = 0.9524\ldots$ – M. Lassak, 1982

Some Results in $\mathbb{R}^3$

1. $d_1^3 = d_2^3 = d_3^3 = 1$
2. $d_4^3 \geq \sqrt{(3+\sqrt{3})/6} = 0.888\ldots$
3. $d_4^3 = \sqrt{(3+\sqrt{3})/6}$ – D. Gale conjecture, 1953
4. $d_4^3 \leq 0.9977$ – A. Heppes, 1957
5. $d_4^3 \leq 0.9887$ – B. Grünbaum, 1957
6. $d_4^3 \leq 0.98$ – V. Makeev, L. Evdokimov, 1997
7. $d_5^3 \leq \sqrt{(35+\sqrt{73})/48} = 0.9524\ldots$ – M. Lassak, 1982

Some Results in $\mathbb{R}^3$

1. $d_1^3 = d_2^3 = d_3^3 = 1$
2. $d_4^3 \geq \sqrt{(3+\sqrt{3})/6} = 0.888\ldots$
3. $d_4^3 = \sqrt{(3+\sqrt{3})/6}$ – D. Gale conjecture, 1953
4. $d_4^3 \leq 0.9977$ – A. Heppes, 1957
5. $d_4^3 \leq 0.9887$ – B. Grünbaum, 1957
6. $d_4^3 \leq 0.98$ – V. Makeev, L. Evdokimov, 1997
7. $d_5^3 \leq \sqrt{(35+\sqrt{73})/48} = 0.9524\ldots$ – M. Lassak, 1982

Some Results in $\mathbb{R}^3$

1. $d_1^3 = d_2^3 = d_3^3 = 1$
2. $d_4^3 \geq \sqrt{(3+\sqrt{3})/6} = 0.888\ldots$
3. $d_4^3 = \sqrt{(3+\sqrt{3})/6}$ – D. Gale conjecture, 1953
4. $d_4^3 \leq 0.9977$ – A. Heppes, 1957
5. $d_4^3 \leq 0.9887$ – B. Grünbaum, 1957
6. $d_4^3 \leq 0.98$ – V. Makeev, L. Evdokimov, 1997
7. $d_5^3 \leq \sqrt{(35+\sqrt{73})/48} = 0.9524\ldots$ – M. Lassak, 1982

Some Results in $\mathbb{R}^3$

1. $d_1^3 = d_2^3 = d_3^3 = 1$
2. $d_4^3 \geq \sqrt{(3+\sqrt{3})/6} = 0.888\ldots$
3. $d_4^3 = \sqrt{(3+\sqrt{3})/6}$ – D. Gale conjecture, 1953
4. $d_4^3 \leq 0.9977$ – A. Heppes, 1957
5. $d_4^3 \leq 0.9887$ – B. Grünbaum, 1957
6. $d_4^3 \leq 0.98$ – V. Makeev, L. Evdokimov, 1997
7. $d_5^3 \leq \sqrt{(35+\sqrt{73})/48} = 0.9524\ldots$ – M. Lassak, 1982

main result

Theorem 1

The inequality holds $d_5^3 \leq 0.9425$.

Definitions, main property

- U is a *universal cover* in $\mathbb{R}^n$, if

$$\forall\ \Phi \subset \mathbb{R}^n,\ \operatorname{diam}\Phi = 1,\ \exists\ O \in Eucl(\mathbb{R}^n),\ \ O(U) \supseteq \Phi$$

- $\mathcal{U}$ is a *universal covering system* in $\mathbb{R}^n$, if

$$\forall\ \Phi \subset \mathbb{R}^n,\ \operatorname{diam}\Phi = 1,\ \exists\ U \in \mathcal{U},$$

$$\exists\ O \in Eucl(\mathbb{R}^n),\ \ O(U) \supseteq \Phi$$

- $\forall\ \mathcal{U}$ – universal covering system in $\mathbb{R}^n$

$$d_k^n \leq \sup_{U \in \mathcal{U}} d_k^n(U).$$

Definitions, main property

- U is a *universal cover* in $\mathbb{R}^n$, if

$$\forall\ \Phi \subset \mathbb{R}^n,\ \operatorname{diam}\Phi = 1,\ \exists\ O \in Eucl(\mathbb{R}^n),\ \ O(U) \supseteq \Phi$$

- $\mathcal{U}$ is a *universal covering system* in $\mathbb{R}^n$, if

$$\forall\ \Phi \subset \mathbb{R}^n,\ \operatorname{diam}\Phi = 1,\ \exists\ U \in \mathcal{U},$$
$$\exists\ O \in Eucl(\mathbb{R}^n),\ \ O(U) \supseteq \Phi$$

- $\forall\ \mathcal{U}$ – universal covering system in $\mathbb{R}^n$

$$d_k^n \le \sup_{U \in \mathcal{U}} d_k^n(U).$$

Definitions, main property

- U is a *universal cover* in $\mathbb{R}^n$, if

$$\forall\ \Phi \subset \mathbb{R}^n,\ \operatorname{diam}\Phi = 1,\ \exists\ O \in Eucl(\mathbb{R}^n),\ \ O(U) \supseteq \Phi$$

- $\mathcal{U}$ is a *universal covering system* in $\mathbb{R}^n$, if

$$\forall\ \Phi \subset \mathbb{R}^n,\ \operatorname{diam}\Phi = 1,\ \exists\ U \in \mathcal{U},$$
$$\exists\ O \in Eucl(\mathbb{R}^n),\ \ O(U) \supseteq \Phi$$

- $\forall\ \mathcal{U}$ – universal covering system in $\mathbb{R}^n$

$$d_k^n \le \sup_{U \in \mathcal{U}} d_k^n(U).$$

Examples of universal covers

1. Ball with radius r, $r = \sqrt{\frac{n}{2n+2}}$ in $\mathbb{R}^n$, H. Jung, 1903
2. (Lassak, 1982) $U = B_1 \cap B_2$ in $\mathbb{R}^n$,

$$B_1 = \{X = (x_1, \ldots, x_n) : \; x_1^2 + \ldots + x_n^2 \leq r^2\},$$

$$B_2 = \{X = (x_1, \ldots, x_n) : \; (x_1 - r)^2 + x_2^2 + \ldots + x_n^2 \leq 1\}.$$

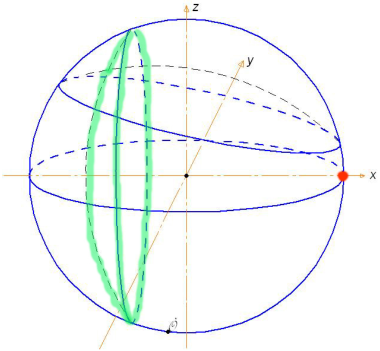
Examples of universal covers

1. Ball with radius r, $r = \sqrt{\frac{n}{2n+2}}$ in $\mathbb{R}^n$, H. Jung, 1903
2. (Lassak, 1982) $U = B_1 \cap B_2$ in $\mathbb{R}^n$,

$$B_1 = \{X = (x_1, \ldots, x_n) : \; x_1^2 + \ldots + x_n^2 \le r^2\},$$
$$B_2 = \{X = (x_1, \ldots, x_n) : \; (x_1 - r)^2 + x_2^2 + \ldots + x_n^2 \le 1\}.$$

Universal covering system

$r = \sqrt{3/8}$ – radius of Jung ball in $\mathbb{R}^3$, $\quad d \in [0.5, r]$
$D = D(d) = \{X = (x, y, z) : \; x^2 + y^2 + z^2 \leq d^2\}, \; S = \partial D$
$D_1 = \{X = (x, y, z) : \; (x - d)^2 + y^2 + z^2 \leq 1\}.$

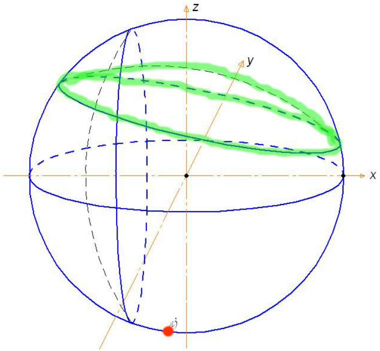


Universal covering system

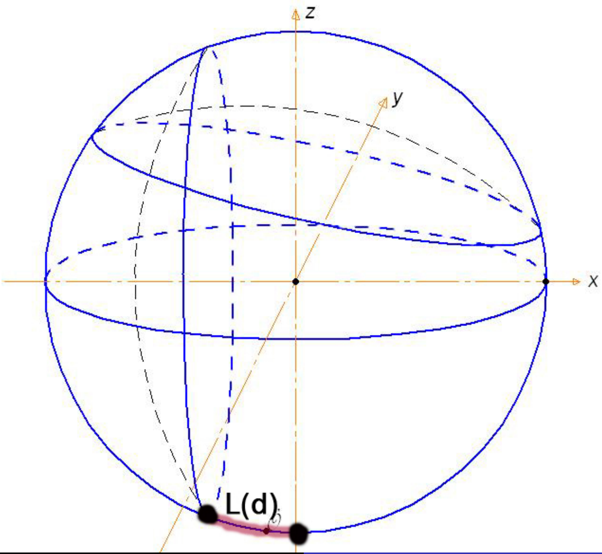
$$\mathcal{O} = (a, 0, c),$$

$\mathcal{O} \in L(d) = S \cap D_1 \cap \{X = (x, y, z) : \ x \leq 0\} \cap \{X = (x, y, z) : \ y = 0\},$

$D_2 = D_2(\mathcal{O}) = \{X = (x, y, z) : \ (x - a)^2 + y^2 + (z - c)^2 \leq 1\}$ – ball with radius 1 and center in $\mathcal{O}$

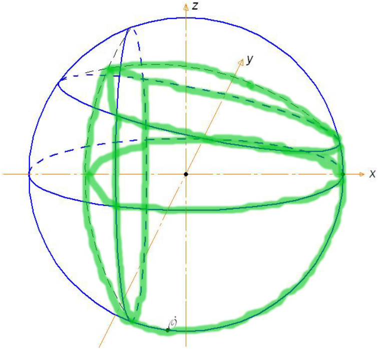


Universal covering system



Universal covering system

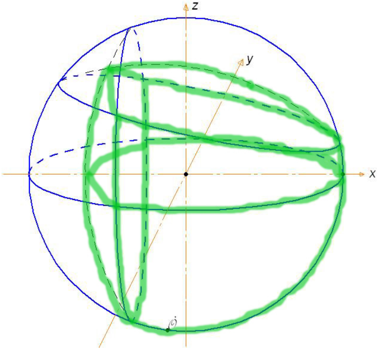
- Universal covering system $\mathcal{U}$:
$$\mathcal{U} = \{U(d, \mathcal{O}) : \ d \in [0.5, r], \ \mathcal{O} \in L(d)\}$$
$\mathcal{U}$ is a set family with continuum cardinality, where
$$U(d, \mathcal{O}) = D \cap D_1 \cap D_2.$$



- **Lemma 1** $\mathcal{U}$ forms a universal covering system in $\mathbb{R}^3$

Universal covering system

- Universal covering system $\mathcal{U}$:
$$\mathcal{U} = \{U(d, \mathcal{O}) : \ d \in [0.5, r], \ \mathcal{O} \in L(d)\}$$
$\mathcal{U}$ is a set family with continuum cardinality, where
$$U(d, \mathcal{O}) = D \cap D_1 \cap D_2.$$

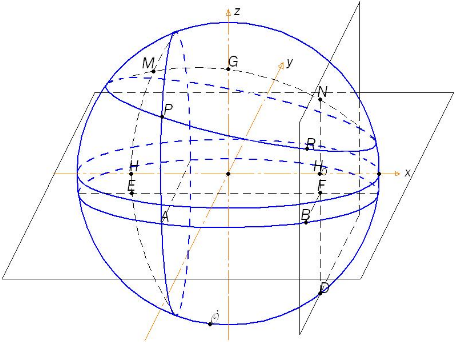


- **Lemma 1** $\mathcal{U}$ forms a universal covering system in $\mathbb{R}^3$

Construction of partitioning

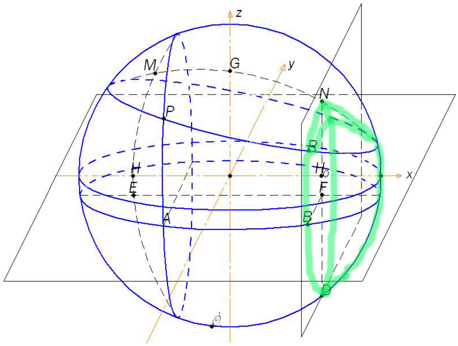
$\forall U \in \mathcal{U}$ we construct a partitioning into 5 parts subject to variables $\varepsilon, \delta \in (0, d)$:

$$P_1 = P_1(U, \varepsilon, \delta), \; \ldots, \; P_5 = P_5(U, \varepsilon, \delta)$$



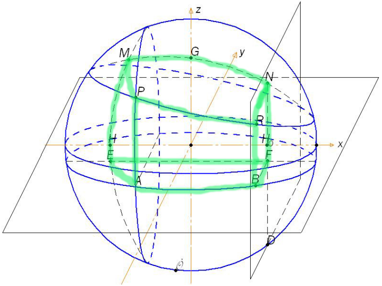
Construction of partitioning

$$P_1 = U \cap \{X = (x, y, z) : \; x \geq r - \varepsilon\},$$



Construction of partitioning

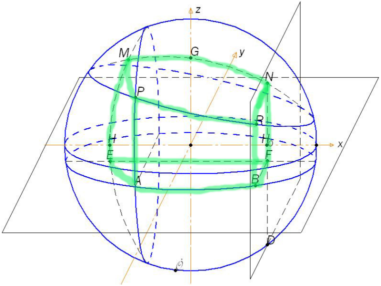
- $P_2 = U \cap \{X = (x, y, z) : \; x \leq r - \varepsilon\} \cap$ $\{X = (x, y, z) : \; z \geq -\delta\} \cap \{X = (x, y, z) : \; y \leq 0\}$,



- P_4 symmetrical P_2 with respect to plane $\{X = (x, y, z) : \; y = 0\}$.

Construction of partitioning

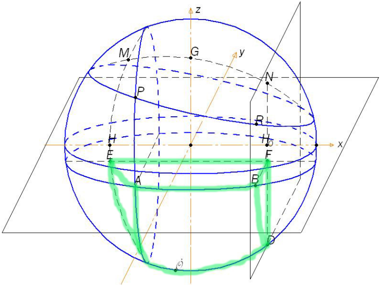
- $P_2 = U \cap \{X = (x, y, z) : \ x \le r - \varepsilon\} \cap \{X = (x, y, z) : \ z \ge -\delta\} \cap \{X = (x, y, z) : \ y \le 0\}$,



- P_4 symmetrical P_2 with respect to plane $\{X = (x, y, z) : \ y = 0\}$.

Construction of partitioning

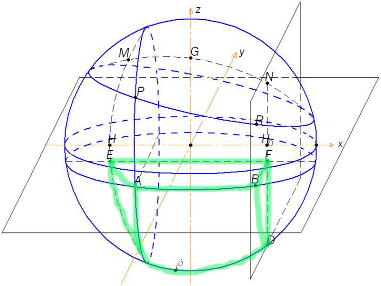
- $P_3 = U \cap \{X = (x,y,z): \ x \le r - \varepsilon\} \cap \{X = (x,y,z): \ z \le -\delta\} \cap \{X = (x,y,z): \ y \le 0\}$;



- P_5 symmetrical P_3 with respect to plane $\{X = (x,y,z): \ y = 0\}$.

Construction of partitioning

- $P_3 = U \cap \{X = (x, y, z) : \ x \leq r - \varepsilon\} \cap \{X = (x, y, z) : \ z \leq -\delta\} \cap \{X = (x, y, z) : \ y \leq 0\};$



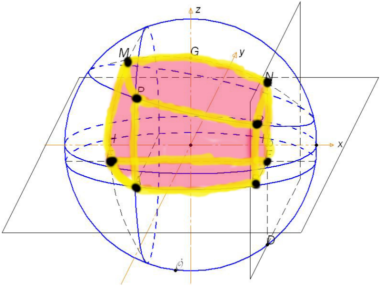
- P_5 symmetrical P_3 with respect to plane $\{X = (x, y, z) : \ y = 0\}$.

Scetch of proof

- first case: $d \geq 0.592$

  Lemma 2 Diameter of P_2 is attained on points from $\partial^1 P_2 \cup \partial^0 P_2$. Similarly for P_3.

  Lemma 3 Diameter of P_2 is attained on points from $\partial^0 P_2$. Similarly for P_3.



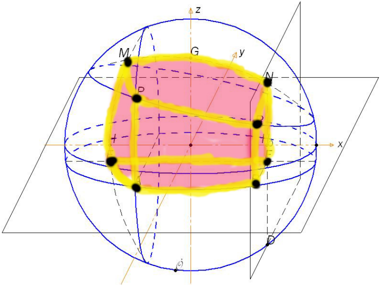
- second case $d \leq 0.592$ – is similar to Lassak's construction

Scetch of proof

- first case: $d \geq 0.592$

  Lemma 2 Diameter of P_2 is attained on points from $\partial^1 P_2 \cup \partial^0 P_2$. Similarly for P_3.

  Lemma 3 Diameter of P_2 is attained on points from $\partial^0 P_2$. Similarly for P_3.



- second case $d \leq 0.592$ – is similar to Lassak's construction

=)

Thank You