## On dividing three-dimensional sets into five parts of smaller diameter

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## Borsuk's problem

- $f(\Omega)=\min \left\{f: \Omega=\Omega_{1} \cup \ldots \cup \Omega_{f}, \operatorname{diam} \Omega_{i}<\operatorname{diam} \Omega\right\}$

- $f(n)=n+1 \quad$ K. Borsuk's conjecture, 1933
- $f(n)=n+1$ for $n \leq 3$ $f(n)>n+1$ for $n \geq 298$
- $(1.2255 \ldots+o(1))^{\sqrt{n}} \leq f(n) \leq(1.224 \ldots+o(1))^{n}$


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## Definition of $d_{k}^{n}$

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d_{k}^{n}(\Phi)=\inf \left\{x \geq 0: \Phi=\Phi_{1} \cup \Phi_{2} \cup \ldots \cup \Phi_{k}, \operatorname{diam} \Phi_{i} \leq x\right\}
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d_{k}^{n}=\sup _{\Phi, \operatorname{diam} \Phi=1} d_{k}^{n}(\Phi)
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## Results in $\mathbb{R}^{2}$

(1) $d_{2}^{1}=d_{2}^{2}=1$
(3) $d_{3}^{2}=\frac{\sqrt{3}}{2}$

- $d_{4}^{2}=\frac{1}{\sqrt{2}}-H$. Lenz, 1956
- $d_{7}^{2}=\frac{1}{2}-$ H. Lenz, 1956
- $d_{k}^{2}$ for many other cases - V. Filimonov, 2008


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## Some Results in $\mathbb{R}^{3}$

(1) $d_{1}^{3}=d_{2}^{3}=d_{3}^{3}=1$

$d_{4}^{3} \leq 0.9977-A$. Heppes, 1957


- B. Grünbaum, 1957
© $d_{4}^{3} \leq 0.98$ - V. Makeev, L. Evdokimov, 1997
- $d_{5}^{3} \leq \sqrt{(35+\sqrt{73}) / 48}=0.9524 \ldots-$ M. Lassak, 1982


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- $d_{4}^{3}=\sqrt{(3+\sqrt{3}) / 6-D . \text { Gale conjecture, } 1953}$
- $d_{4}^{3} \leq 0.9977$ - A. Heppes, 1957
- $d_{4}^{3} \leq 0.9887$ - B. Grünbaum, 1957
- $d_{4}^{3} \leq 0.98-$ V. Makeev, L. Evdokimov, 1997
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## main result

Theorem 1
The inequality holds $d_{5}^{3} \leq 0.9425$.

## Definitions, main property

- U is a universal cover in $\mathbb{R}^{n}$, if
$\forall \Phi \subset \mathbb{R}^{n}, \operatorname{diam} \Phi=1, \exists O \in \operatorname{Eucl}\left(\mathbb{R}^{n}\right), O(U) \supseteq \Phi$
- $\mathcal{U}$ is a universal covering system in $\mathbb{R}^{n}$, if
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d_{k}^{n} \leq \sup _{U \in \mathcal{U}} d_{k}^{n}(U)
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## Examples of universal covers

(1) Ball with radius $r, r=\sqrt{\frac{n}{2 n+2}}$ in $\mathbb{R}^{n}$, H. Jung, 1903 (3) (Lassak, 1982) $U=B_{1} \cap B_{2}$ in $\mathbb{R}^{n}$,


## Examples of universal covers

(1) Ball with radius $r, r=\sqrt{\frac{n}{2 n+2}}$ in $\mathbb{R}^{n}$, H. Jung, 1903
(2) (Lassak, 1982) $U=B_{1} \cap B_{2}$ in $\mathbb{R}^{n}$,

$$
\begin{gathered}
B_{1}=\left\{X=\left(x_{1}, \ldots, x_{n}\right): x_{1}^{2}+\ldots+x_{n}^{2} \leq r^{2}\right\} \\
B_{2}=\left\{X=\left(x_{1}, \ldots, x_{n}\right):\left(x_{1}-r\right)^{2}+x_{2}^{2}+\ldots+x_{n}^{2} \leq 1\right\} .
\end{gathered}
$$

## Universal covering system

$$
\begin{aligned}
& r=\sqrt{3 / 8}-\text { radius of Jung ball in } \mathbb{R}^{3}, \quad d \in[0.5, r] \\
& D=D(d)=\left\{X=(x, y, z): x^{2}+y^{2}+z^{2} \leq d^{2}\right\}, S=\partial D \\
& D_{1}=\left\{X=(x, y, z):(x-d)^{2}+y^{2}+z^{2} \leq 1\right\} .
\end{aligned}
$$



## Universal covering system

$$
\begin{aligned}
& \mathcal{O}=(a, 0, c), \\
& \mathcal{O} \in L(d)=S \cap D_{1} \cap\{X=(x, y, z): x \leq 0\} \cap\{X= \\
& (x, y, z): y=0\}, \\
& D_{2}=D_{2}(\mathcal{O})=\left\{X=(x, y, z):(x-a)^{2}+y^{2}+(z-c)^{2} \leq 1\right\}- \\
& \text { ball with radius } 1 \text { and center in } \mathcal{O}
\end{aligned}
$$



## Universal covering system



## Universal covering system

- Universal covering system $\mathcal{U}$ :

$$
\mathcal{U}=\{U(d, \mathcal{O}): d \in[0.5, r], \mathcal{O} \in L(d)\}
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$\mathcal{U}$ is a set family with continuum cardinality, where

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U(d, \mathcal{O})=D \cap D_{1} \cap D_{2}
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- Lemma $1 \mathcal{U}$ forms a universal covering system in $\mathbb{R}^{3}$

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## Construction of partitioning

$\forall U \in \mathcal{U}$ we construct a partitioning into 5 parts subject to variables $\varepsilon, \delta \in(0, d)$ :

$$
P_{1}=P_{1}(U, \varepsilon, \delta), \ldots, P_{5}=P_{5}(U, \varepsilon, \delta)
$$



## Construction of partitioning

$$
P_{1}=U \cap\{X=(x, y, z): x \geq r-\varepsilon\},
$$



## Construction of partitioning

- $P_{2}=U \cap\{X=(x, y, z): x \leq r-\varepsilon\} \cap$

$$
\{X=(x, y, z): z \geq-\delta\} \cap\{X=(x, y, z): y \leq 0\}
$$



- $P_{4}$ symmetrical $P_{2}$ with respect to nlane $\{X=(x, y, z): y=0\}$


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- $P_{5}$ symmetrical $P_{3}$ with respect to plane $\{X=(x, y, z): y=0\}$.


## Scetch of proof

- first case: $d \geq 0.592$

Lemma 2 Diameter of $P_{2}$ is attained on points from $\partial^{1} P_{2} \cup \partial^{0} P_{2}$. Similarly for $P_{3}$.
Lemma 3 Diameter of $P_{2}$ is attained on points from $\partial^{0} P_{2}$. Similarly for $P_{3}$.


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- second case $d \leq 0.592$ - is similar to Lassak's construction

Introduction

## Thank You

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